math 262 T3 Take Home - interesting problems

(1) $$\int_2^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx = \int_2^{5} \frac{1}{x\sqrt{x^2-4}}\,dx + \int_5^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx$$

$x \neq 0$, $x \neq 2$ over $[2, \infty)$ — improper I_1 — improper I_2

let $I = \int \frac{1}{x\sqrt{x^2-4}}\,dx$; note : Pg 481

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{x\sqrt{x^2-1}}$$

ie, $$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + C$$

ie, $$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$ by sub

so $$I = \int \frac{1}{x\sqrt{x^2-4}}\,dx = \boxed{\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C}$$

φ $a^2 = 4$

$a = 2$

$$I_1(t) = \lim_{t \to 2^+} \int_t^{5} \frac{1}{x\sqrt{x^2-4}}\,dx = \lim_{t \to 2^+} \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)\Big|_t^{5}$$

$$= \frac{1}{2}\sec^{-1}\left(\frac{5}{2}\right) - \frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right)$$ as $t \to 2^+$

$I_1(t) \to \frac{1}{2}\sec^{-1}(5/2)$

as $\frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right) \to 0$

see class notes!

$$I_2(t) = \int_5^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx = \lim_{t\to\infty} \int_5^{t} \frac{1}{x\sqrt{x^2-4}}\,dx$$

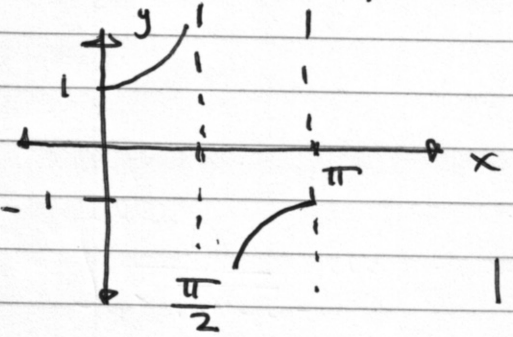
$$= \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)\Big|_5^t = \frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right) - \frac{1}{2}\sec^{-1}(5/2)$$

as $t \to \infty$;

$$I_2(t) = \frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right) - \frac{1}{2}\sec^{-1}(5/2)$$

$$I_2(t) \to \frac{1}{2}\left[\frac{\pi}{2}\right] - \frac{1}{2}\sec^{-1}(5/2)$$

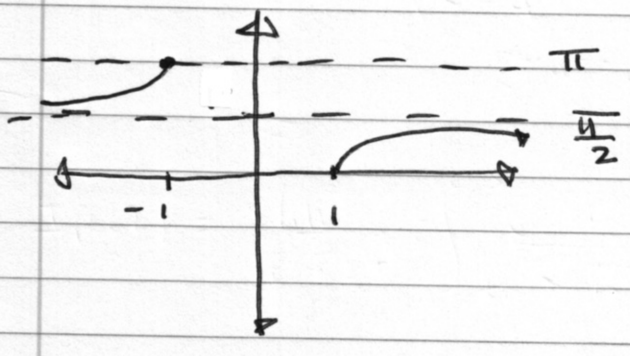
note $y = \sec(x)$; $0 \le x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \le \pi$; $|y| \ge 1$



Restricted Domain
so $y = \sec(x)$ is 1 to 1

Reflect about x-axis

$y = \sec^{-1}(x)$; $|x| \ge 1$; $0 \le y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \le \pi$



ie, $\boxed{\lim_{x\to\infty} \sec^{-1}(x) = \frac{\pi}{2}}$

$I_2(t) = \frac{\pi}{4} - \frac{1}{2}\sec^{-1}(5/2)$; $I(t) \to I_1(t) + I_2(t)$

$\frac{1}{2}\sec^{-1}(5/2) + \frac{\pi}{4} - \frac{1}{2}\sec^{-1}(5/2)$

$I(t) \to \boxed{\frac{\pi}{4}}$ converges

③ $\int_0^\infty f(t)\, e^{-st}\, dt$ where $f(t) = t$

$\text{Dom}(F) = \{ s \mid \underbrace{F(s) < \infty}_{\text{converges}} \}$

$$\int_0^\infty t e^{-st}\, dt = \lim_{t \to \infty} \int_0^t t e^{-st}\, dt$$

maybe we need another variable?

$$I(t) = \int_0^t t e^{-st}\, dt = uv - \int v\, du$$

IBP $= -\frac{t}{s} e^{-st} - \int -\frac{1}{s} e^{-st}\, dt$

$u = t$ $\quad dv = e^{-st}\, dt$

$du = dt$ $\quad v = \int e^{-st}\, dt$ $\quad = -\frac{t}{s} e^{-st} + \frac{1}{s} \cdot \int e^{-st} dt$

$v = -\frac{1}{s} e^{-st}$ $\quad = -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \Big|_0^t$

$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}$$

$$- \left[-\frac{0}{s} e^{-s \cdot 0} - \frac{1}{s^2} e^{-s \cdot 0} \right]$$

$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} + \frac{1}{s^2}$$

$$I(t) = -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} + \frac{1}{s^2}$$

__note__ if $s < 0$, then $-s > 0$ and $I(t) \to \infty$ as $t \to \infty$

if $s < 0$, $I(t) \to \frac{1}{s^2}$ converges

$$I(t) = -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} + \frac{1}{s^2}$$

$$\lim_{t\to\infty}\left(\cancel{-\frac{t}{s}e^{-st}}^{\,0} - \cancel{\frac{1}{s^2}e^{-st}}^{\,0} + \frac{1}{s^2}\right)$$

as $t\to\infty$; $e^{-st}\to 0$; so $-\frac{1}{s^2}e^{-st} \to \boxed{0}$

$$\therefore \lim_{t\to\infty} -te^{-st} = \lim_{t\to\infty} \frac{-t}{e^{st}} \quad \frac{\infty}{\infty}$$

$$\underset{H}{=} \lim_{t\to\infty} \frac{-1}{se^{st}} = \lim_{t\to\infty} -\frac{1}{s}e^{-st} = \boxed{0}$$