East Los Angeles College
Department of Mathematics
Math 262
Test 3

42

Evaluate the following improper integrals.

1. $\int_{-1}^{1} \frac{e^x}{e^x-1}\,dx$

$$= \underbrace{\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx}_{I_1} + \underbrace{\int_{0}^{1} \frac{e^x}{e^x-1}\,dx}_{I_2}$$

VA $x=0$;

$I = \int \frac{e^x}{e^x-1}\,dx$; $u = e^x-1$, $du = e^x\,dx$

$$= \int \frac{1}{u}\,du = \ln|u| = \ln|e^x-1| + C$$

$$I_1 = \int_{-1}^{0} \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^x}{e^x-1}\,dx$$

$$= \ln|e^x-1| \Big|_{x=-1}^{x=t} = \ln|e^t-1| - \ln|e^{-1}-1|$$

$$= \ln|e^t-1| - \ln\left(\frac{1}{e}-1\right) \text{ ; as } t\to 0^-$$

$|e^t-1| \to \cancel{+\infty}\ 0$

$\ln|e^t-1| \to -\infty$

Div

ie, don't have to check I_2 ; $\boxed{\text{Divergence}}$

6

4. Find the area of the surface obtained by rotating the curve about the **x-axis**.

$$y = \frac{x^3}{6} + \frac{1}{2x} \text{ for } \frac{1}{2} \le x \le 1$$

$$SA = \int 2\pi \, r \, ds$$

$$r = y \qquad ds = \sqrt{1 + (y')^2} \, dx$$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$\sqrt{1 + (y')^2} = \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2} = \left| \frac{1}{2}x^2 + \frac{1}{2x^2} \right|$$

$$= \frac{1}{2}x^2 + \frac{1}{2x^2}$$

$$SA = \int_{1/2}^{1} 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right) dx$$

$$= 2\pi \int_{1/2}^{1} \left(\frac{x^5}{12} + \frac{1}{3}x + \frac{1}{4}x^{-3}\right) dx$$

$$= 2\pi \left[\frac{1}{72}x^6 + \frac{1}{6}x^2 - \frac{1}{8x^2} \right]_{1/2}^{1}$$

6✓

$$\boxed{\frac{263\pi}{256}}$$

3. Set Up-Use Simpson's rule with n=6 to estimate the arc length for the following curve.

$$y = x\ln x \text{ for } 1 \le x \le 3$$

$y' = 1 + \ln x$;

$$S = \int_1^3 \underbrace{\sqrt{1 + (1+\ln x)^2}}_{f(x)}\, dx$$

$\Delta x = \frac{3-1}{6} = \frac{1}{3}$; $n = 6$

$S \approx S_6$; $S_6 = \frac{1}{9}\Big[f(1) + 4f(4/3)$

$+ 2f(\frac{5}{3}) + 4f(2)$

$+ 2f(7/3) + 4f(8/3)$

$+ f(3)\Big]$

$f(x) = \sqrt{1 + (1+\ln x)^2}$

6✓

2. Determine the arc length for the following curve over the indicated interval.

$$y = \ln(\cos x) \text{ for } 0 \le x \le \pi/3$$

$y' = -\tan x$; $1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$

$$s = \int_0^{\pi/3} \sqrt{1+(y')^2}\, dx = \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/3} |\sec x|\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx$$

$$= \ln|\sec x + \tan x| \Big|_{x=0}^{x=\pi/3}$$

$$= \ln|\sec(\tfrac{\pi}{3}) + \tan(\pi/3)|$$

$$- \ln|\sec(0) + \tan(0)|$$

$$= \ln(\sqrt{3} + 2) \ ;$$

$$\boxed{s = \ln(\sqrt{3}+2)}$$

6v

5. Find the area of the surface obtained by rotating the curve about the **y-axis**.

$$y = 1 - x^2 \text{ for } 0 \le x \le 1$$

$$SA = \int_0^1 2\pi \underset{x}{r}\, \underset{\sqrt{1+(y')^2}\,dx}{ds}$$

$$y' = -2x \;; \quad \sqrt{1+(y')^2} = \sqrt{1+4x^2}$$

$$SA = \int_0^1 2\pi x \sqrt{1+4x^2}\,dx \;, \quad u\text{-sub}$$

$$u = 1+4x^2$$

$$\frac{du}{dx} = 8x$$

$$dx = \frac{du}{8x}$$

$$SA = \int_1^5 2\pi \cancel{x} \sqrt{u}\, \frac{du}{8\cancel{x}}$$

$$SA = \int_1^5 \frac{\pi}{4} u^{1/2}\,du = \frac{\pi}{4}\int_1^5 u^{1/2}\,du$$

$$= \frac{\pi}{4} \frac{u^{3/2}}{3/2}\Bigg|_{u=1}^{u=5} = \frac{\pi}{6} u^{3/2}\Bigg|_{u=1}^{u=5}$$

$$= \frac{\pi}{6}\left[5^{3/2} - 1^{3/2}\right]$$

$$= \frac{\pi}{6}\left[5\sqrt{5} - 1\right]$$

6✓

6. Determine the equation of the line tangent to the curve at the indicated point.

$$x = e^{-t}$$
$$y = 1 - cos(\pi t)$$
$$t = 1$$

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\pi \sin(\pi t)}{e^{-t}(-1)}$$

$$m = \frac{\pi \sin(\pi t)}{e^{-t}} = \pi e^{t} \sin(\pi t)$$

ie, $m = \pi e^{1} \cancel{\sin}^{0} (\pi \cdot 1)$; $m = 0$

$$y - y_1 = m(x - x_1) ;$$

$$y - y_1 = 0(x - x_1) ; \quad y - y_1 = 0$$

$$y = y_1 ; \quad y_1 = 1 - \cos(\pi t) \Big|_{t=1}$$

$$y_1 = 1 - \cos(\pi)$$

$$y_1 = 1 - (-1)$$

$$y_1 = 2$$

6✓

ie, $\boxed{y = 2}$

7. Find points on the curve where the tangent is horizontal.

$$x = 10 - t^2$$
$$y = t^3 - 12t$$

$\frac{dy}{dt} = 0$; $3t^2 - 12 = 0$;

$t = \pm 2$

$t = 2$; $x = 10 - 2^2 = 6$

$y = 2^3 - 12 \cdot 2 = -16$

$\boxed{(6, -16)}$

$t = -2$; $x = 10 - (-2)^2 = 6$

$y = (-2)^3 - 12(-2) = 16$

$\boxed{(6, 16)}$

6 ✓

math 262 Test 3

(1) $\displaystyle\int_{-1}^{1} \frac{e^x}{e^x-1}\,dx$

note $e^x - 1 = 0$; $e^x = 1$;

$\ln(e^x) = \ln(1)$; $x\ln(e) = 0$

$x = 0$; $\boxed{VA}$

$$\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx + \int_{0}^{1} \frac{e^x}{e^x-1}\,dx$$

let $I = \displaystyle\int \frac{e^x}{e^x-1}\,dx$; $u = e^x - 1$

$du = e^x\,dx$

$dx = \dfrac{du}{e^x}$

$$= \int \frac{\cancel{e^x}}{u} \cdot \frac{du}{\cancel{e^x}} = \int \frac{1}{u}\,du$$

$$= \ln|u| = \ln|e^x - 1| + C$$

(I) $\displaystyle\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^x}{e^x-1}\,dx$

$$= \lim_{t\to 0^-} \ln|e^x - 1| \Big|_{-1}^{t}$$

$$= \lim_{t\to 0^-} \left[\ln|e^t - 1| - \ln|e^{-1} - 1|\right]$$

as $t \to 0^-$; $e^t \to 1$; $e^t - 1 \to 0$

$\ln|e^t - 1| \to \boxed{-\infty}$

Div

(2) $y = \ln(\cos x)$; $0 \le x \le \frac{\pi}{3}$

$y' = \frac{1}{\cos x}(-\sin x) = -\tan x$

$(y')^2 = (-\tan x)^2 = \tan^2 x$

$\int \sqrt{1+(y')^2}\,dx = \int \sqrt{1+\tan^2 x}\,dx$

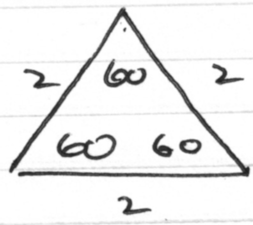
$= \int \sqrt{\sec^2 x}\,dx = \int |\sec x|\,dx = \int \sec x\,dx$

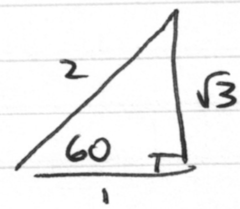
$= \ln|\sec x + \tan x| \Big|_0^{\pi/3}$

$= \ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right|$

$- \ln|\sec(0) + \tan(0)|$

note





$\sec\left(\frac{\pi}{3}\right) = 2$; $\sec(0) = \frac{1}{\cos(0)} = 1$

$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\tan(0) = 0$

$s = \ln|2+\sqrt{3}| - \cancel{\ln|1+0|}^{\,0}$

$\boxed{s = \ln(\sqrt{3}+2)}$

(4) $y = \frac{x^3}{6} + \frac{1}{2x}$; $\frac{1}{2} \le x \le 1$

Revolve abut
x- axis !

$$SA = \int 2\pi \underset{y}{r}\, ds \quad \sqrt{1+(y')^2}\, dx$$

$$y' = \frac{3x^2}{6} - \frac{1}{2x^2} \; ; \quad y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$1 + (y')^2 = 1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2$$

$$= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

$$\sqrt{1+(y')^2} = \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} = \left|\frac{x^2}{2} + \frac{1}{2x^2}\right|$$

$$= \frac{x^2}{2} + \frac{1}{2x^2} \quad \text{as } \frac{1}{2} \le x \le 1$$

$$\text{ie, } SA = \int_{1/2}^{1} 2\pi \left[\frac{x^3}{6} + \frac{1}{2x}\right]\left[\frac{x^2}{2} + \frac{1}{2x^2}\right] dx$$

foil

$$= 2\pi \int_{1/2}^{1} \frac{x^5}{12} + \frac{1}{12}x + \frac{1}{4}x + \frac{1}{4x^3}\, dx$$

$$SA = \int_{1/2}^{1} \left(\frac{1}{12} x^5 + \frac{1}{3} x + \frac{1}{4} x^{-3} \right) dx$$

$$\frac{1}{12 \cdot 6} x^6 + \frac{1}{3 \cdot 2} x^2 + \frac{1}{4(-2)} x^{-2} \Big|_{1/2}^{1}$$

$$\frac{1}{72} x^6 + \frac{1}{6} x^2 - \frac{1}{8x^2} \Big|_{1/2}^{1}$$

$\cancel{\frac{1}{47}}$

$$\frac{1}{72} + \frac{1}{6} - \frac{1}{8} - \left(\frac{1}{72} \left(\frac{1}{2}\right)^6 + \frac{1}{6} \left(\frac{1}{2}\right)^2 - \frac{1}{8 (1/2)^2} \right)$$

$$\frac{1}{72} + \frac{12}{12 \cdot 6} - \frac{9}{72} - \left(\frac{1}{72} \cdot \frac{1}{64} + \frac{1}{6} \cdot \frac{1}{4} - \frac{1}{2} \right)$$

$$\frac{1}{72} + \frac{12}{72} - \frac{9}{72} - \left(\frac{1}{72 \cdot 64} + \frac{1}{24} - \frac{1}{2} \right)$$

$$\frac{13-9}{72} - \left(\frac{1}{4608} + \frac{1}{24} - \frac{1}{2} \right)$$

$$\frac{4}{72} - \frac{1}{4608} - \frac{1}{24} + \frac{1}{2}$$

$$\frac{1}{18} - \frac{1}{4608} - \frac{1}{24} + \frac{1}{2}$$

$$\frac{256 - 1 - 192 + 2304}{4608}$$

$$\left| \frac{2367}{4608} \right| ; \quad 2\pi \frac{2367}{4608}$$

$$\frac{4734\pi}{4608}$$

$$\boxed{\frac{263\pi}{256}}$$

(5) $y = 1 - x^2 \ ; \ 0 \leq x \leq 1$

y-axis

$$SA = \int 2\pi \, r \, ds$$

$$r = x, \quad ds = \sqrt{1 + (y')^2} \, dx$$

$y = 1 - x^2 \ ; \quad y' = -2x \ ; \quad (y')^2 = (-2x)^2$

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4x^2}$$

$$SA = \int_0^1 2\pi x \sqrt{1 + 4x^2} \, dx$$

$$= 2\pi \int_0^1 x \sqrt{1 + 4x^2} \, dx$$

u-sub; $u = 1 + 4x^2$, $dx = \frac{du}{8x}$

$du = 8x \, dx$

$$SA = \int_1^5 2\pi \cancel{x} \sqrt{u} \frac{du}{8\cancel{x}}$$

$$= \frac{\pi}{4} \int_1^5 u^{1/2}\, du$$

$$= \frac{\pi}{4} \frac{u^{3/2}}{3/2} \Big|_{u=1}^{u=5}$$

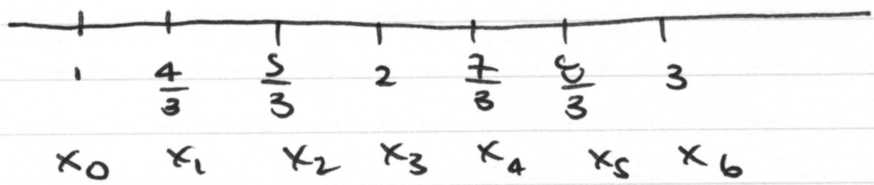
$$= \frac{\pi}{4} \cdot \frac{2}{3} \; u^{3/2} \Big|_{u=1}^{u=5}$$

$$\frac{\pi}{6} \left[5^{3/2} - 1^{3/2} \right]$$

$$\boxed{\frac{\pi}{6} \left[5\sqrt{5} - 1 \right]}$$

(3) $n = 6$; $1 \le x \le 3$

$$\Delta x = \frac{3-1}{6} = \frac{1}{3}$$



$y = x \ln x$; $y' = x \cdot \frac{1}{x} + \ln x$

$y' = 1 + \ln x$

$$s = \int_1^3 \overbrace{\sqrt{1 + (1+\ln x)^2}}^{f(x)} \, dx$$

Simpsons Rule: S_6

$$S_6 = \frac{1/3}{3} \left[f(x_0) + 4 f(x_1) + 2 f(x_2) \right.$$

$$+ 4 f(x_3) + 2 f(x_4) +$$

$$\left. + 4 f(x_5) + f(x_0) \right]$$

i.e. $S_6 = \frac{1}{9} \left[f(1) + 4 f\left(\frac{4}{3}\right) + 2 f\left(\frac{5}{3}\right) \right.$

$+ 4 f(2) + 2 f\left(\frac{7}{3}\right) + 4 f(8/3)$

$\left. + f(3) \right]$

$f(x) = \sqrt{1 + (1 + \ln x)^2}$

(b) $y - y_1 = m(x - x_1)$

$x = e^{-1}$; $y = 1 - \cos(\pi)$

as $t = 1$

$x = \frac{1}{e}$; $y = 1 - -1$

$y = 2$

$\left(\overset{x}{\frac{1}{e}}, \overset{y}{2}\right)$,

$$m = \frac{dy/dt}{dx/dt} = \frac{-\sin(\pi t)\cdot\pi}{e^{-t}(-1)}$$

$$m = \frac{\pi \sin(\pi t)}{e^{-t}} = e^{t}\pi \sin(\pi t)$$

$$m = \pi e^{t} \sin(\pi t) \Big|_{t=1}$$

$m = \pi e^{1} \cancel{\sin}^{0}(\pi)$; $m = 0$

$y - y_1 = 0$; $\boxed{y = 2}$

(7) $\frac{dy}{dt} = 0$; $3t^2 - 12 = 0$

$3(t^2 - 4) = 0$

$t^2 - 4 = 0$

$t^2 = 4$; $t = \pm\sqrt{4}$

$t = \pm 2$

$t = 2$; $x = 10 - 2^2 = 10 - 4 = 6$

$y = 2^3 - 12 \cdot 2 = 8 - 24 = -16$

$(6, -16)$

$t = -2$; $x = 10 - (-2)^2 = 10 - 4 = 6$

$y = (-2)^3 - 12(-2) = -8 + 24 = 16$

$(6, 16)$