East Los Angeles College

Department of Mathematics

Math 262

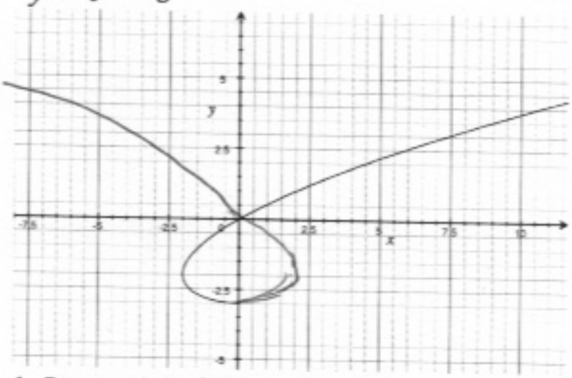
Test 4 In Class

38

Solutions

$x = t^3 - 3t$

$y = t^2 - 3$



1. Determine the horizontal tangents
2. Determine the vertical tangents.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-3}$$

HT

$dy/dt = 0$

$2t = 0$

$t = 0$

$\boxed{(0, -3)}$

8

VT

$dx/dt = 0$

$3t^2 - 3 = 0$

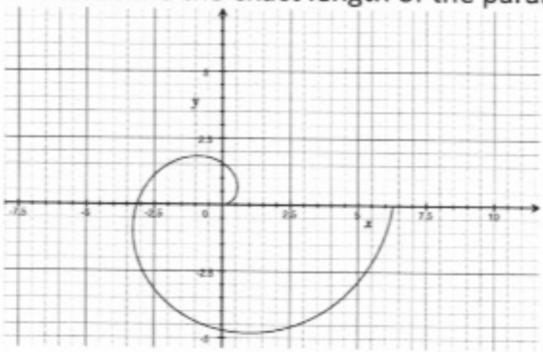
$t = \pm 1$; $t = 1$

$t = 1$; $\boxed{(-2, -2)}$

$t = 1$; $\boxed{(2, -2)}$

$x = t\cos(t)$
$y = t\sin(t)$
$0 \le t \le 1$

3. Determine the exact length of the parametric curve.



$$S = \int_0^1 \sqrt{(x')^2 + (y')^2}\, dt$$

$x' = \cos(t) - t\sin(t)$ ✓

$y' = \sin(t) + t\cos(t)$ ✓

$\sqrt{(x')^2 + (y')^2} = \sqrt{1 + t^2}$

$$S = \int_0^1 \sqrt{1+t^2}\, dt$$ ✓

Trig sub

$t = \tan\theta$; $dt = \sec^2\theta \, d\theta$

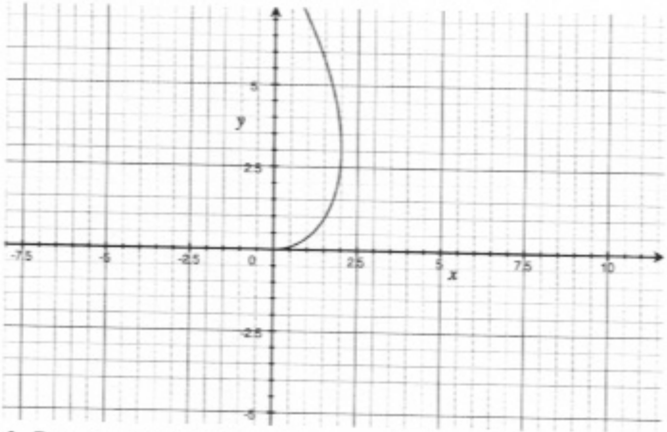
$\sqrt{1+t^2} = \sec\theta$ ✓

$$S = \int_0^{\pi/4} \sec^3\theta \, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]\Big|_{\theta=0}^{\theta=\frac{\pi}{4}}$$ ✓

$$\boxed{\frac{\sqrt{2}}{2} + \ln(\sqrt{2}+1)}$$ ✓ ✓

7 ✓

$x = 3t - t^3$
$y = 3t^2$
$0 \le t \le 1$



$SA = \int 2\pi r \, ds$

$r \to y \to 3t^2$

4. Determine the exact area of the surface by rotating the parametric curve about the x-axis.

$$dS = \sqrt{(x')^2 + (y')^2}$$

$$x' = 3 - 3t^2 \; ; \; y' = 6t$$

$$ds = \sqrt{(x')^2 + (y')^2}$$

$$= 3 + 3t^2$$

$$SA = \int_0^1 2\pi \cdot 3t^2 \, (3 + 3t^2) \, dt$$

$$= 6\pi \int_0^1 (3t^2 + 3t^4) \, dt$$

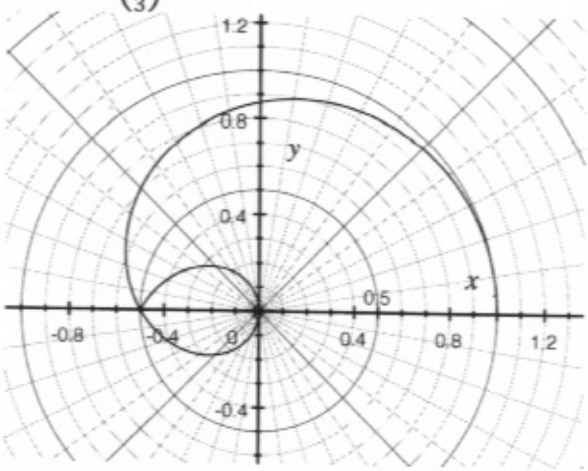
$$= 18\pi \int_0^1 (t^2 + t^4) \, dt$$

$$= 18\pi \left[\tfrac{1}{3} t^3 + \tfrac{1}{5} t^5 \Big|_0^1 \right]$$

$$= \boxed{\frac{48\pi}{5}}$$

6

$r = cos\left(\frac{\theta}{3}\right)$ and $\theta = \pi$



$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$ ✓

$$\frac{dr}{d\theta} = -\frac{1}{3}\sin\left(\frac{\theta}{3}\right)$$

✓

5. Determine the slope of the tangent line to the polar curve.

✓

$$\frac{dy}{dx} = \left.\frac{-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\sin\theta + \cos\left(\frac{\theta}{3}\right)\cos\theta}{-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\cos\theta - \cos\left(\frac{\theta}{3}\right)\sin\theta}\right|_{\theta=\pi}$$

$$= \frac{-\frac{1}{3}\sin\left(\frac{\pi}{3}\right)\overset{0}{\cancel{\sin}}(\pi) + \cos\left(\frac{\pi}{3}\right)\overset{-1}{\cancel{\cos}}(\pi)}{-\frac{1}{3}\sin\left(\frac{\pi}{3}\right)\underset{-1}{\cos(\pi)} - \cos\left(\frac{\pi}{3}\right)\overset{0}{\cancel{\sin(\pi)}}}$$

S✓

$$= \frac{-\cos(\pi/3)}{\frac{1}{3}\sin(\pi/3)}$$

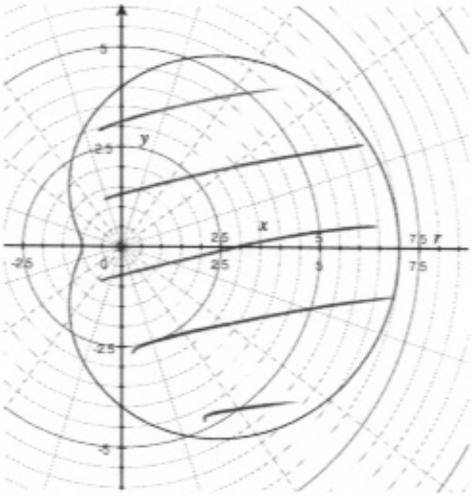
$$= -3\cot(\pi/3)$$ ✓

$$= -3\frac{\sqrt{3}}{3}$$

✓

$$= \boxed{-\sqrt{3}}$$

$r = 4 + 3cos(\theta)$



$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta$$

$$A = \int_0^{2\pi} \frac{1}{2} (4+3\cos(\theta))^2 d\theta$$

$$A = 2\int_0^{\pi} \frac{1}{2} (4+3\cos(\theta))^2 d\theta$$

✓

6. Determine the area enclosed by the polar curve.

$$A = \int_0^{\pi} (4+3\cos(\theta))^2 d\theta \quad ✓$$

$$= \int_0^{\pi} (16 + 24\cos\theta + 9\cos^2(\theta)) d\theta \quad ✓$$

$$= \int_0^{\pi} \frac{41}{2} + 24\cos\theta + \frac{9}{2}\cos(2\theta)\, d\theta \quad ✓$$

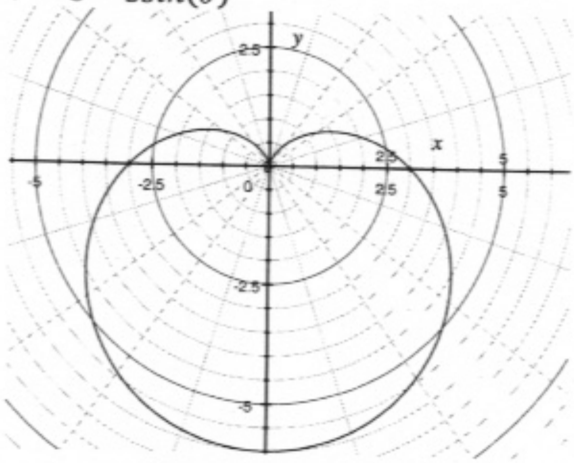
$$\frac{41}{2}\theta + 24\sin\theta + \frac{9}{4}\sin(2\theta) \Big|_0^{\pi}$$

✓

$$\boxed{\frac{41\pi}{2}} \quad ✓$$

6✓

$r = 3 - 3sin(\theta)$



$$S = \int_0^{2\pi} \sqrt{r^2 + (r')^2}\, d\theta$$

$$r' = -3\cos\theta$$

7. Find the exact length of the polar curve.

$$\sqrt{r^2+(r')^2} = \sqrt{18(1-\sin\theta)} = \sqrt{18}\sqrt{1-\sin\theta} \quad ✓$$

$$= 3\sqrt{2}\sqrt{1-\sin\theta} \cdot \frac{\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta}}$$

$$= \frac{3\sqrt{2}\,|\cos\theta|}{\sqrt{1+\sin\theta}} ;$$

$$S = \int_0^{2\pi} \frac{3\sqrt{2}\,|\cos\theta|}{\sqrt{1+\sin\theta}}\, d\theta = 2\int_{-\pi}^{\pi} 3\sqrt{2}\,\frac{\cos\theta}{\sqrt{1+\sin\theta}}\, d\theta$$

$$= 6\sqrt{2} \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1+\sin\theta}}\, d\theta ; \quad u\text{-sub}$$

✓ ✓ ✓

$$= 6\sqrt{2}\int u^{-1/2}\, du = 12\sqrt{2}\sqrt{u}$$

6✓

$$= 12\sqrt{2}\sqrt{1+\sin\theta}\,\Big|_{-\pi/2}^{\pi/2} \quad ✓$$

$$= 12\sqrt{2}\sqrt{1+1} - 12\sqrt{2}\sqrt{1-1}$$

$$= \boxed{24} \quad ✓$$

math 262 Test 4 (class)

1

$x = t^3 - 3t$
$y = t^2 - 3$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-3}$$

Horizontal Tangents ; $\frac{dy}{dx} = 0$ or $2t = 0$

$t = 0$

$x = 0^3 - 3 \cdot 0$
$x = 0$ $\quad \boxed{(0,-3)}$
$y = 0^2 - 3$
$y = -3$

Vertical Tangents

$3t^2 - 3 = 0$; $3t^2 = 3$; $t^2 = 1$; $t = \pm\sqrt{1}$

$t = \pm 1$

$t = 1$; $x = 1^3 - 3 \cdot 1$; $y = 1^2 - 3$
$x = 1 - 3$
$x = -2$ $\quad y = 1 - 3$ $\quad \boxed{(-2,-2)}$
$y = -2$

$t = -1$; $x = (-1)^3 - 3(-1)$
$y = (-1)^2 - 3$
$x = -1 + 3$
$x = 2$ $\quad y = 1 - 3$ $\quad \boxed{(2,-2)}$
$y = -2$

(3) $x = t\cos(t)$

$y = t\sin(t)$

$0 \le t \le 1$

$$S = \int_0^1 \sqrt{(x')^2 + (y')^2}\, dt$$

$$x' = \frac{d}{dt}(t\cos(t)) = t\,\cancel{\frac{d}{dt}}\overset{-\sin(t)}{(\cos(t))} + \cos(t)\overset{1}{\cancel{\frac{dt}{dt}}}$$

$$= \cos(t) - t\sin(t)$$

$$y' = \frac{d}{dt}(t\sin t) = t\,\cancel{\frac{d}{dt}}\overset{\cos(t)}{(\sin t)} + \sin t\,\overset{1}{\cancel{\frac{dt}{dt}}}$$

$$= \sin t + t\cos(t)$$

$$(x')^2 + (y')^2 = (\cos t - t\sin t)^2$$

$$+ (\sin t + t\cos t)^2$$

$$= (\cos t - t\sin t)(\cos t - t\sin t)$$

$$+ (\sin t + t\cos t)(\sin t + t\cos t)$$

$$= \cancel{\cos^2 t} - \cancel{t\sin t\cos} - t\sin t\cos t + t^2\sin^2 t$$

$$+ \cancel{\sin^2 t} + \cancel{t\sin t\cos t} + t\sin t\cos t + t^2\cos^2 t$$

$$= 1 + t^2 \; ; \; \sqrt{(x')^2 + (y')^2} = \sqrt{1+t^2}$$

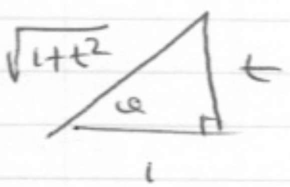
$$\int_0^1 \sqrt{1+t^2}\, dt \quad ; \text{ use trig sub}$$

$$t = \tan\theta$$
$$dt = \sec^2\theta \, d\theta$$
$$\sqrt{1+t^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta}$$
$$= \sec\theta$$

$$\int \sec\theta \sec^2\theta \, d\theta = \int \sec^3\theta \, d\theta$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]$$

$$\tan\theta = \frac{t}{1} \quad ; \quad \tan\theta = t$$



$$\cos\theta = \frac{1}{\sqrt{1+t^2}} \quad ; \sec\theta = \sqrt{1+t^2}$$

$$= \frac{1}{2}\left[\sqrt{1+t^2}\, t + \ln\left|\sqrt{1+t^2} + t\right|\right]$$

$$\frac{1}{2}\left[t\sqrt{1+t^2} + \ln\left|\sqrt{1+t^2} + t\right|\right]\Bigg|_{t=0}^{t=1}$$

$$\frac{1}{2}\left[1\sqrt{1+1^2} + \ln\left|\sqrt{1+1^2} + 1\right|\right]$$

$$- \frac{1}{2}\left[0\sqrt{1+0^2} + \ln\left|\sqrt{1+0^2} + 0\right|\right]$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln|\sqrt{2}+1|\right] - \frac{1}{2}\ln|1| \; 0$$

$$\boxed{\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(\sqrt{2}+1) - \cancel{\frac{1}{2}\ln(1)}}$$

(4) $x = 3t - t^3$

$y = 3t^2$ $\qquad$ $SA = \int 2\pi\, r\, ds$

$0 \le t \le 1$

$$\begin{array}{c} r \\ \Downarrow \\ y \\ \Downarrow \\ 3t^2 \end{array}$$

$ds = \sqrt{(x')^2 + (y')^2}$

$x' = 3 - 3t^2 \quad ; \quad y' = 6t$

$$\begin{aligned}
(x')^2 + (y')^2 &= (3-3t^2)^2 + (6t)^2 \\
&= (3-3t^2)(3-3t^2) + 36t^2 \\
&= 9 - 9t^2 - 9t^2 + 9t^4 + 36t^2 \\
&= 9 - 18t^2 + 9t^4 + 36t^2 \\
&= 9 + {+18}t^2 + 9t^4 \\
&= (3+3t^2)(3+3t^2) \\
&= (3+3t^2)^2
\end{aligned}$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{(3+3t^2)^2} = |3+3t^2|$$

$$= 3 + 3t^2$$

$$SA = \int_0^1 2\pi \cdot 3t^2\,(3+3t^2)\,dt$$

$$= 6\pi \int_0^1 (3t^2 + 3t^4)\,dt$$

$$SA = 6\pi \left[\frac{3t^3}{3} + \frac{3t^5}{5} \Big|_0^1 \right]$$

$$SA = 6\pi \left[t^3 + \frac{3}{5} t^5 \Big|_0^1 \right]$$

$$SA = 6\pi \left[1^3 + \frac{3}{5} 1^5 \right] - \cancel{6\pi \left[0^3 + \frac{3}{5} 0^5 \right]}^{\,0}$$

$$SA = 6\pi \left(1 + \frac{3}{5} \right) = 6\pi \left(\frac{5}{5} + \frac{3}{5} \right)$$

$$= 6\pi \, \frac{8}{5} = \boxed{\frac{48\pi}{5}}$$

(5) $r = \cos\left(\frac{\theta}{3}\right)$; $\theta = \pi$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{dr}{d\theta} = -\sin\left(\frac{\theta}{3}\right) \cdot \frac{1}{3} = -\frac{1}{3}\sin\left(\frac{\theta}{3}\right)$$

$$\frac{dy}{dx} = \frac{-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\sin\theta + \cos\left(\frac{\theta}{3}\right)\cos\theta}{-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\cos\theta - \cos\left(\frac{\theta}{3}\right)\sin\theta} \Bigg|_{\theta=\pi}$$

$$\frac{dy}{dx} = \frac{-\frac{1}{3}\sin\left(\frac{\pi}{3}\right)\cancel{\sin(\pi)}^{0} + \cos\left(\frac{\pi}{3}\right)\cancel{\cos(\pi)}^{-1}}{-\frac{1}{3}\sin\left(\frac{\pi}{3}\right)\cancel{\cos(\pi)}^{-1} - \cos\left(\frac{\pi}{3}\right)\cancel{\sin(\pi)}^{0}}$$

$$= \frac{-\cos\left(\frac{\pi}{3}\right)}{+\frac{1}{3}\sin\left(\frac{\pi}{3}\right)} = -3\cot\left(\frac{\pi}{3}\right)$$

$$= \frac{-3}{\sqrt{3}} = \frac{-3}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = -\sqrt{3}$$

(6) $r = 4 + 3\cos\varphi \; ; \quad 0 \le \varphi \le 2\pi$

$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\varphi$$

$$(4+3\cos\varphi)$$

ie, $$A = 2\int_0^{\pi} \frac{1}{2}(4+3\cos\varphi)^2 \, d\varphi$$

Symmetry

$$= \int_0^{\pi} (4+3\cos\varphi)(4+3\cos\varphi)\,d\varphi$$

$$= \int_0^{\pi} 16 + 12\cos\varphi + 12\cos\varphi + 9\cos^2\varphi \, d\varphi$$

$$= \int_0^{\pi} 16 + 24\cos\varphi + 9\underbrace{\cos^2\varphi}_{\frac{1}{2}(1+\cos(2\varphi))} \, d\varphi$$

$$= \int_0^{\pi} 16 + 24\cos\varphi + \frac{9}{2}(1+\cos(2\varphi))\,d\varphi$$

$$\frac{2\cdot 16}{2} + 24\cos\varphi + \frac{9}{2} + \frac{9}{2}\cos(2\varphi)$$

$$\int \frac{32}{2} + \frac{9}{2} + 24\cos\varphi + \frac{9}{2}\cos(2\varphi)\,d\varphi$$

$$\int \frac{41}{2} + 24\cos\varphi + \frac{9}{2}\cos(2\varphi)\,d\varphi$$

$$\frac{41}{2}\varphi + 24\sin\varphi + \frac{9}{4}\sin(2\varphi)\Big|_0^{\pi} \qquad \boxed{\frac{41\pi}{2}}$$

$$\frac{41\pi}{2} + 24\underbrace{\sin(\pi)}_{0} + \frac{9}{4}\underbrace{\sin(2\pi)}_{0}$$

$$- 0 - 24\underbrace{\sin(0)}_{0} - \frac{9}{4}\underbrace{\sin(0)}_{0}$$

(7) $r = 3 - 3\sin\theta$

$r' = -3\cos\theta$

$$r^2 + (r')^2 = (3 - 3\sin\theta)^2 + (-3\cos\theta)^2$$

$$= (3-3\sin\theta)(3-3\sin\theta) + 9\cos^2\theta$$

$$= 9 - 9\sin\theta - 9\sin\theta + 9\sin^2\theta + 9\cos^2\theta$$

$$= 9 - 18\sin\theta + 9\,\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}$$

$$= 9 - 18\sin\theta + 9$$

$$= 18 - 18\sin\theta \,;$$

$$\sqrt{r^2 + (r')^2} = \sqrt{18 - 18\sin\theta} = \sqrt{18(1-\sin\theta)}$$

$$= \sqrt{18}\,\sqrt{1-\sin\theta} = 3\sqrt{2}\,\sqrt{1-\sin\theta}$$

ie, $$S = \int_0^{2\pi} 3\sqrt{2}\,\sqrt{1-\sin\theta}\;d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\sqrt{2}\,\sqrt{1-\sin\theta}\;d\theta$$

$$= 6\sqrt{2}\int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{1-\sin\theta}\;d\theta$$

$$6\sqrt{2}\int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{1-\sin\theta}\ \frac{\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta}}\ d\theta$$

$$6\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1+\sin\theta}}\ d\theta$$

$$6\sqrt{2}\int_{-\frac{\pi}{2}}^{\pi/2} \frac{|\cos\theta|}{\sqrt{1+\sin\theta}}\ d\theta \ ;$$

$|\cos\theta| = \cos\theta$ over $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$6\sqrt{2}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1+\sin\theta}}\ d\theta$$

u-sub

$u = 1+\sin\theta$

$du = \cos\theta\ d\theta$

$d\theta = \dfrac{du}{\cos\theta}$

$$6\sqrt{2}\int_{-\pi/2}^{\pi/2} \frac{\cancel{\cos\theta}}{\sqrt{u}}\ \frac{du}{\cancel{\cos\theta}}$$

$$6\sqrt{2}\int_{-\pi/2}^{\pi/2} u^{-1/2}\ du = 6\sqrt{2}\ \frac{u^{1/2}}{1/2}$$

$$= 12\sqrt{2}\ \sqrt{1+\sin\theta}\ \Big|_{-\pi/2}^{\pi/2}$$

$$12\sqrt{2}\sqrt{1+\sin(\pi/2)} - 12\sqrt{2}\sqrt{1+\sin(-\pi/2)}$$

$$12\sqrt{2}\sqrt{1+1} - 12\sqrt{2}\sqrt{1-\sin(\pi/2)}$$

$$12\sqrt{2}\sqrt{2} - 12\sqrt{2}\sqrt{1-1}$$

$$12\cdot 2 - \cancel{12\sqrt{2}\sqrt{0}}^{\,0}$$

$$\boxed{24}$$