East Los Angeles College
Department of Mathematics 6S ✓
Math 262
Test 2 Class

Evaluate the following limit.

1. $\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$ 1^∞ (See Scratch)

5 ✓

$\ln y = x \ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)$ ✓

$\ln y = \dfrac{\ln\left(1+\frac{3}{x}+\frac{3}{x^2}\right)}{1/x}$ ✓ ; as $x\to\infty$

$\ln y \to 3$ ✓

$y \to e^3$ ✓

Integrate the following.

2. $\int x^2 \sin(\pi x)\,dx$ IBP (See Scratch) 5 ✓

$-\frac{1}{\pi}x^2\cos(\pi x) + \frac{2}{\pi^2}x\sin(\pi x) + \frac{2}{\pi^3}\cos(\pi x) + C$

5 ✓

3. $\int sec(x)dx$ $= \ln|\sec x + \tan x| + C$

15 ✓

4. $\int \sin^2(x)dx$ $= \int \frac{1}{2}[1 - \cos(2x)]\, dx$

$= \int \frac{1}{2}\, dx - \frac{1}{2}\int \cos(2x)\, dx$

$\boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$

5/

5. $\int \sin^2(x)\cos^3(x)$dx $= \int \sin^2 x\,(1 - \sin^2(x))\cos(x)\, dx$

$u = \sin x$

$\int u^2 - u^4 \; du$

$\frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

5/

$\boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}$

10/

10/

12. $\int_1^\infty \frac{\ln x}{x^2}dx$ $= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx$ (See Scratch)

$I = \int_1^t \frac{\ln x}{x^2} = -\frac{\ln x}{x} - \frac{1}{x}$; IBP

as $x \to -\infty$; $-\frac{1}{x} \to 0$

$-\frac{\ln x}{x} \to 0$; ie $I \to 0$

$\boxed{\text{Converges}}$

5✓

13. $\int_{-\infty}^\infty \frac{x^2}{9+x^6}dx$ $= \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx + \int_0^\infty \frac{x^2}{9+x^6}\,dx$ 5✓

$I_1 = \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{x^2}{9+x^6}\,dx$

$= -\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) \to \frac{\pi}{18}$ as $x \to \infty$

$I_2 = \int_0^\infty \frac{x^2}{9+x^6}\,dx = \lim_{t\to\infty} \int_0^t \frac{x^2}{9+x^6}\,dx$

$= \frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) \to \frac{\pi}{18}$ as $x \to \infty$

ie, $\frac{\pi}{9}$ Convergence (See Scratch) 10✓

$$① \lim_{x\to\infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x \qquad 1^\infty$$

$$y = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x ; \quad \ln y = \ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$$

$$\text{or} \quad \ln y = x \cdot \ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right) ;$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \left[x \ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)\right] \qquad \infty \cdot 0$$

$$\text{ie,} \quad \ln y = \frac{\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{1/x} \qquad \text{as } x \to \infty \quad \frac{0}{0}$$

$$-\frac{3}{x^2} - \frac{10}{x^3}$$

$$\underset{H}{=} \lim_{x\to\infty} \frac{\dfrac{1}{1 + \frac{3}{x} + \frac{5}{x^2}} \quad \frac{d}{dx}\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{-\dfrac{1}{x^2}}$$

$$\text{note} \quad \frac{1}{1 + \frac{3}{x} + \frac{5}{x^2}} \left(-\frac{3}{x^2} - \frac{10}{x^3}\right) \div \left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{1 + \frac{3}{x} + \frac{5}{x^2}} \left(-\frac{3}{x^2} - \frac{10}{x^3}\right)\left(-x^2\right)$$

$$= \frac{1}{1 + \frac{3}{x} + \frac{5}{x^2}} \left(3 + \frac{10}{x}\right)$$

(2) $\int x^2 \sin(\pi x)\, dx$; IBP

$u = x^2 \quad dv = \sin(\pi x)\, dx$

$\frac{du}{dx} = 2x \qquad v = \int \sin(\pi x)\, dx$

$du = 2x\, dx$

$v = -\frac{1}{\pi} \cos(\pi x)$

$uv - \int v\,du = -\frac{1}{\pi} x^2 \cos(\pi x)$

$- \int -\frac{1}{\pi} \cos(\pi x)\, 2x\, dx$

$-\frac{1}{\pi} x^2 \cos(\pi x) + \frac{2}{\pi} \int x \cos(\pi x)\, dx$

let $u = x$; $dv = \cos(\pi x)\, dx$

$du = dx$

$v = \int \cos(\pi x)\, dx$

$v = \frac{1}{\pi} \sin(\pi x)$

IBP $uv - \int v\,du$

$\frac{1}{\pi} x \sin(\pi x) - \int \frac{1}{\pi} \sin(\pi x)\, dx$

$\frac{1}{\pi} x \sin(\pi x) - \frac{1}{\pi} \left(\int \sin(\pi x)\, dx \right)$

$\left[-\frac{1}{\pi} \cos(\pi x) \right)$

(6) $\int \tan^4(x)\,dx = \int \tan^2 x\,(\sec^2 x - 1)\,dx$

$\tan^2 x \cdot \tan^2 x$

$\sec^2 x - 1$

$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$

ie, $\int \tan^4(x)\,dx = \underbrace{\int \tan^2 x \sec^2 x}_{I_1} - \underbrace{\int \tan^2 x\,dx}_{I_2}$

$I_1 = \int \tan^2 x \sec^2 x\,dx$; $u = \tan x$

$du = \sec^2 x\,dx$

$dx = \frac{du}{\sec^2 x}$

$= \int u^2 \cdot \cancel{\sec^2 x}\,\frac{du}{\cancel{\sec^2 x}} = \int u^2\,du$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}\tan^3 x + C$

$I_2 = \int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$

$\sec^2 x - 1$

$= \tan x - x + C$

$\boxed{\frac{1}{3}\tan^3 x - \tan x + x + C}$

(7) $\int \frac{\sin(2x)\cos(5x)}{4}\, dx$

$\frac{1}{2}\left[\sin(7x) + \sin(-3x)\right]$

$\frac{1}{2}\left[\sin(7x) - \sin(3x)\right]$

$\frac{1}{2}\sin(7x) - \frac{1}{2}\sin(3x)$

$\int \frac{1}{2}\sin(7x) - \frac{1}{2}\sin(3x) \quad dx$

$\frac{1}{2}\int \sin(7x)\, dx - \frac{1}{2}\int \sin(3x)\, dx$

$-\frac{1}{2}\cdot\frac{1}{7}\cos(7x) - \frac{1}{2}\left(-\frac{1}{3}\right)\cos(3x)\, \cancel{dx}$

$\boxed{-\frac{1}{14}\cos(7x) + \frac{1}{6}\cos(3x) + C}$

(8) $\int \frac{\sqrt{1+x^2}}{x}\, dx$; $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$\sqrt{1+x^2} = \sec\theta$

$\int \frac{\sec\theta}{\tan\theta}\sec^2\theta\, d\theta = \int \frac{\sec^3\theta}{\tan\theta}\, d\theta$

$\int \frac{\sec\theta\sec^2\theta}{\tan\theta}\, d\theta = \int \frac{\sec\theta(\tan^2\theta + 1)}{\tan\theta}$

$= \int \frac{\sec\theta\tan^2\theta + \sec\theta}{\tan\theta}\, d\theta$

now, $\dfrac{\sec\theta \tan^2\theta}{\tan\theta} + \dfrac{\sec\theta}{\tan\theta}$

$= \sec\theta \tan\theta + \cancel{\dfrac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}} \quad \dfrac{1}{\sin\theta}$

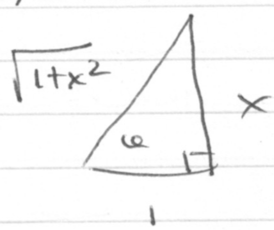
$\sec\theta$

ie, $\int \left(\sec\theta \tan\theta + \sec\theta\right) d\theta$

$\int \sec\theta \tan\theta \, d\theta + \int \sec\theta \, d\theta$

$\sec\theta + \ln|\sec\theta + \tan\theta|$

But, $x = \tan\theta$; $\tan\theta = \dfrac{x}{1}$



$\cos\theta = \dfrac{1}{\sqrt{1+x^2}}$

$\sec\theta = \sqrt{1+x^2}$

ie, $\boxed{\sqrt{1+x^2} + \ln\left|x + \sqrt{1+x^2}\right| + C}$

(9) $\int \sqrt{5+4x-x^2}\, dx$

note $-x^2+4x+5 = -1\cdot(x^2-4x-5)$

or $\dfrac{-x^2+4x+5}{-1} = x^2-4x-5$

① $-\dfrac{4}{2} = -2$

② $(-2)^2 = 4$

$\dfrac{-x^2+4x+5}{-1} = \underline{x^2-4x+4}-4-5$

$= \cancel{x+}\ (x-2)^2 - 9$

or $-x^2-4x+5 = -(x-2)^2+9$

$= 9-(x-2)^2$

$\int \sqrt{5+4x-x^2}\, dx = \int \sqrt{9-(x-2)^2}\, dx$

let $u = x-2$; $du = dx$

$= \int \sqrt{9-u^2}\, du$ Trig Sub

$u = 3\sin\theta$

$du = 3\cos\theta\, d\theta$

$\sqrt{9-u^2} = 3\cos\theta$

$$\int \sqrt{9-u^2}\, du = \int 3\cos\theta \cdot 3\cos\theta \, d\theta$$

$$= 9 \int \cos^2\theta \, d\theta$$

$$\tfrac{1}{2}[1+\cos(2\theta)]$$

$$= \frac{9}{2}\int [1+\cos(2\theta)]\, d\theta$$

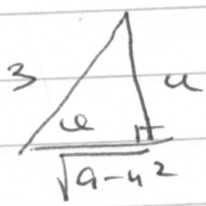
$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]$$

$$\frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C$$

$$2\sin\theta\cos\theta$$

$$\frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C$$

But, $u = 3\sin\theta$; $\sin\theta = \frac{u}{3}$; $\theta = \sin^{-1}\left(\frac{u}{3}\right)$



$$\cos\theta = \frac{\sqrt{9-u^2}}{3}$$

$$\frac{9}{2}\sin^{-1}\left(\frac{u}{3}\right) + \frac{9}{2}\,\frac{u}{3}\,\frac{\sqrt{9-u^2}}{3} + C$$

$$\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{9-(x-2)^2} + C$$

(10) $\int \frac{2x+3}{(x+1)^2}\,dx$ pfd

$$\frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$\rightarrow$ $2x+3 = A(x+1) + B$

$2x+3 = Ax + A + B$

so, $A=2$ $A+B=3$; $2+B=3$

$B=1$

ie, $\int \frac{2}{x+1} + \frac{1}{(x+1)^2}\,dx$

$$= 2\int \frac{1}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx$$

u-sub

$u = x+1$; $du = dx$

$\int u^{-2}\,du$

$-\frac{1}{u}$

$$\boxed{2\ln|x+1| - \frac{1}{x+1} + C}$$

(11) $\int \frac{10}{(x-1)(x^2+9)}\,dx$ pfd

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$\rightarrow \quad 10 = A(x^2+9) + (x-1)(Bx+C)$$

let $x=1$; $10 = A(1^2+9)$ or

$10 = 10A$; $A=1$

also, $10 = A\cancel{x^2} + 9A + B\cancel{x^2} + C\cancel{x} - B\cancel{x} - C$

$$10 = (A+B)x^2 + (C-B)x + 9A - C$$

$9A - C = 10$; $9 - C = 10$; $C = -1$

$C - B = 0$; $-1 - B = 0$; $B = -1$

$$\int \frac{1}{x-1} + \frac{-x-1}{x^2+9}\,dx$$

$$\int \frac{1}{x-1} - \frac{x+1}{x^2+9}\,dx$$

$$\int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

u = Sub

$\ln|x-1|$ $\qquad u = x^2+9$ $\qquad \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$

$\frac{du}{dx} = 2x$;

$dx = \frac{du}{2x}$

$$\int \frac{\cancel{x}}{u}\,\frac{du}{2\cancel{x}}$$

$\frac{1}{2}\int \frac{1}{u}\,du \qquad \frac{1}{2}\ln|u|$

$\frac{1}{2}\ln|x^2+9|$

$$\ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

(12) $\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx$

$I = \int_1^t \frac{\ln x}{x^2}\,dx$; $\qquad u = \ln x$

$du = \frac{1}{x}\,dx$

$uv - \int v\,du$ $\qquad dv = \int \frac{1}{x^2}\,dx$

$-\frac{\ln x}{x} - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx$ $\qquad v = \int x^{-2}\,dx$

$v = -\frac{1}{x}$

$-\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$

$$I = \frac{-\ln x}{x} + \int x^{-2}\,dx$$

$$\frac{-\ln x}{x} - \frac{1}{x} + C \Big|_1^t$$

ie, $I(t)$

$$\frac{-\ln t}{t} - \frac{1}{t} - \left(-\frac{\ln 1}{1} - \frac{1}{1}\right)$$

$$I = -\frac{\ln t}{t} - \frac{1}{t} + 1 ;$$

$$I(t) = -\frac{\ln t}{t} - \frac{1}{t} + 1$$

as $x \to \infty$; $I(t) \to \boxed{1}$

as $\lim_{t \to \infty} -\frac{\ln t}{t} = 0$

(13) $\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx$

$$= \int_{-\infty}^{0} \frac{x^2}{9+x^6}\,dx + \int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx$$

$$I = \int \frac{x^2}{9+x^6}\,dx$$

$(x^3)^2$

$$= \int \frac{x^2}{9+(x^3)^2}\,dx \quad ; \quad u = x^3$$

$du = 3x^2\,dx$

$dx = \frac{du}{3x^2}$

$$= \int \frac{\cancel{x^2}}{9+u^2}\,\frac{du}{3\cancel{x^2}} = \frac{1}{3}\int \frac{1}{9+u^2}\,du$$

$$= \frac{1}{3}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)$$

$$= \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right)$$

$$I = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)$$

$$\int_{-\infty}^{0} \frac{x^2}{9+x^6}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{x^2}{9+x^6}\,dx$$

so, $I_1 = \frac{1}{9} \tan^{-1}\left(\frac{x^3}{3}\right) \Big|_t^0$

$= \frac{1}{9} \tan^{-1}\left(\frac{0^3}{3}\right) - \frac{1}{9} \tan^{-1}\left(\frac{t^3}{3}\right)$

$= \frac{1}{9} \cancel{\tan^{-1}(0)}^{0} - \frac{1}{9} \tan^{-1}\left(\frac{t^3}{3}\right)$

$= -\frac{1}{9} \tan^{-1}\left(\frac{t^3}{3}\right)$

as $t \to -\infty$; $\frac{t^3}{3} \to -\infty$

$\tan^{-1}\left(\frac{t^3}{3}\right) \to -\frac{\pi}{2}$

ie $\cancel{I_1} = \cancel{\frac{1}{9} \cdot -\frac{\pi}{2}}$; $\cancel{I_1 = \frac{\pi}{18}}$

$-\frac{1}{9} \tan^{-1}\left(\frac{t^3}{3}\right) \to \frac{\pi}{18}$

and; $\cancel{I_2} =$

$\int_0^\infty \frac{x^2}{9+x^6}\, dx = \lim_{t \to \infty} \int_0^t \frac{x^2}{9+x^6}\, dx$

$I_2 = \int_0^t \frac{x^2}{9+x^6}\, dx = \frac{1}{9} \tan^{-1}\left(\frac{x^3}{3}\right) \Big|_0^t$

$= \frac{1}{9} \tan^{-1}\left(\frac{t^3}{3}\right) - \frac{1}{9} \cancel{\tan^{-1}\left(\frac{0^3}{3}\right)}^{0}$

$= \frac{1}{9} \tan^{-1}\left(\frac{t^3}{3}\right)$

as $t \to \infty$, $\frac{t^3}{2} \to \infty$, $\tan^{-1}\left(\frac{t^3}{3}\right) \to \frac{\pi}{2}$

So I_2

$$\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) \to \frac{\pi}{18}$$

Thus, $$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \frac{\pi}{18} + \frac{\pi}{18}$$

$$= \frac{\pi}{9}$$