11.2 From Text Calculus of PE

Determine the equation of the line tangent to the curve.

(5) $x = e^{\sqrt{t}}$; $y = t - \ln t^2$; $t = 1$

(6) $x = \cos\theta + \sin 2\theta$; $y = \sin\theta + \cos 2\theta$; $\theta = 0$

(5) $\frac{dx}{dt} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}$; $\frac{dy}{dt} = 1 - \frac{2t}{t^2} = 1 - \frac{2}{t}$

$$\frac{dy}{dx} = \frac{1 - 2/t}{e^{\sqrt{t}}/2\sqrt{t}} \cdot \frac{2t}{2t} = \frac{2t - 4}{\sqrt{t}\, e^{\sqrt{t}}} \Big|_{t=1}$$

$$\frac{dy}{dx}\Big|_{t=1} = -\frac{2}{e} \leftarrow m$$

note: what is $P(x_1, y_1)$? ; $x = e^{\sqrt{1}}$; $y = 1 - \ln 1^2$

$P(e, 1)$

$$y - y_0 = m(x - x_0)$$

$y_0 = 1$, $m = -\frac{2}{e}$, $x_0 = e$

$$\boxed{y = -\frac{2}{e}x + 3}$$

(6) $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 2\sin 2\theta}{-\sin\theta + 2\cos\theta}\Big|_{\theta=0}$$

$m = \frac{1}{2}$; $x = 1$; $y = 1$ when $\theta = 0$

$y - 1 = \frac{1}{2}(x - 1)$; $\boxed{y = \frac{1}{2}x + \frac{1}{2}}$

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(7) $x = e^t$; $y = (t-1)^2$; $(1,1)$
By eliminating the parameter.

note $x = e^t \rightarrow t = \ln x$

ie, $y = [\ln x - 1]^2$; $y' = 2[\ln x - 1] \cdot \frac{1}{x} \Big|_{x=1}$

$y'\Big|_{x=1} = -2$; $\boxed{y = -2x+3}$

note : parametric term $\frac{dy}{dx} = \frac{2(t-1)}{e^t} \Big|_{(1,1)}$

what is t ?

$1 = e^t$; $1 = (t-1)^2$

$t = 0$ $t=0$

Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. What values of t is $\frac{dx}{dt} > 0$ concave up ?

(11) $x = 4 + t^2$; $y = t^2 + t^3$

$$\frac{dy}{dx} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{3/2}{2t} = \frac{3}{4t}$$

note $\frac{dy}{dx} > 0$ for $1 + \frac{3}{2}t > 0$; $\frac{3}{2}t > -1$

$t > -2/3$

$$\frac{d^2y}{dx^2} > 0 \; ; \quad \frac{3}{dt} > 0 \quad \text{ie} \quad \frac{1}{t} > 0$$

$- - - \;\; \cup \;\; + + +$ $\frac{1}{t}$ $\therefore$ $t > 0$

0

(14) ex $\quad x = t + \ln t \; ; \quad y = t - \ln t$

$$\frac{dy}{dx} = \frac{1 - 1/t}{1 + 1/t} = \frac{t-1}{t+1} = 1 - \frac{2}{t+1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(1 - \frac{2}{t+1}\right)}{1 + \frac{1}{t}} = \frac{\frac{2}{(t+1)^2}}{\frac{t+1}{t}}$$

$$= \frac{2t}{(t+1)^3} \; ; \quad \text{cu} \quad \frac{2t}{(t+1)^3} > 0$$

sign analysis !

note : $t > 0$ Domain

$\cup \quad \cup \quad + + +$

$-1 \quad 0$

$\boxed{t > 0}$ cu

note $\frac{dy}{dx} > 0 \; ; \quad \frac{t+1-2}{t+1} \quad \therefore \quad \frac{t-1}{t+1} > 0$

$+ + + \;\; \cup \;\; - - - \;\; 0 \;\; + + +$ But $t > 0$

$-1 \quad 1$

$t > 1$

(16) $x = \cos 2t \; ; \; y = \cos t \; ; \; 0 < t < \pi$

$$\frac{dy}{dx} = \frac{-\sin t}{-2\sin 2t} = \frac{1}{4\cos t} \qquad \frac{1}{4}\sec t$$

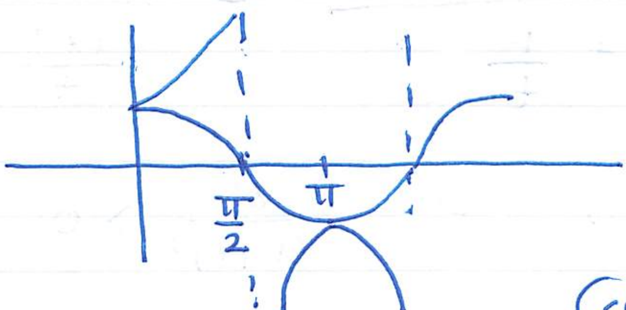
$2\sin t\cos t$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{1}{4}\sec t\right]}{\frac{dx}{dt}} = \frac{\frac{1}{4}\sec t\tan t}{-2\sin(2t)}$$

$-4\sin t\cos t$

$$\frac{d^2y}{dx^2} = \frac{\sec t\tan t}{-16\sin t\cos t} = -\frac{1}{16}\sec^3 t$$

what about $\sec t$?



$\sec t > 0 \; ; \; \cancel{0 \le t <}$
$0 \le t < \frac{\pi}{2}$

ie, $\frac{dy}{dx} > 0 \; ; \; 0 < t < \pi$

(cu) $\sec t < 0$

$-\frac{\pi}{2} < t < \pi$

ie, (cu) $-\frac{\pi}{2} < t < \pi$

Determine points on curve where the tangent is Vertical or Horizontal.

(18) $x = 2t^3 + 3t^2 - 12t \; ; \; y = 2t^3 + 3t^2 + 1$

$$\frac{dy}{dx} = \frac{6t^2 + 6t - 12}{6t^2 + 6t} = \frac{\cancel{6}(t^2 + t - 2)}{\cancel{6}t(t+1)}$$

$$\frac{dy}{dx} = \frac{t^2 + t - 2}{t(t+1)} = \frac{(t+2)(t-1)}{t(t+1)}$$

(19) $\frac{dy}{dx} = \frac{(t+2)(t-1)}{t(t+1)}$

$\frac{dy}{dx} = 0$; $(t+2)(t-1)=0$ or $t=1$, $t=-2$

$\frac{dy}{dx} = \text{und}$; $t=0$; $t+1=0$

$t=-1$

what are the points?

HT VT

$\cancel{t=1}$ $(0,1)$; $(13,2)$ $(20,-3)$, $(-7,6)$

(20) $x = \cos 3\theta$; $y = 2\sin\theta$

$\frac{dy}{dx} = \frac{2\cos\theta}{-3\sin 3\theta}$; $\frac{dy}{dx} = 0$

$2\cos\theta = 0$; $\cos\theta = 0$

$\boxed{\theta = \frac{\pi}{2} + n\pi}$ $(0, \pm 2)$

$\frac{dy}{dx} = \text{und}$;

$-3\sin(3\theta) = 0$ or $\sin(3\theta) = 0$ why?

$\rightarrow 3\theta = n\pi$; or $\boxed{\theta = \frac{\pi}{3}n}$ $(\pm 1, 0)$

$(\pm 1, \pm\sqrt{3})$

(29) At what points on the curve C is the tangent parallel to the line ℓ.

C: $x = t^3 + 4t$; $y = 6t^2$

ℓ: $x = -7t$; $y = 12t - 5$

note : $l = y = 12(-\frac{1}{7}x) - 5$; $m = -\frac{12}{7}$

C: $\frac{dy}{dx} = \frac{12t}{3t^2+4}$; $\cancel{m_t =}$

$\frac{12t}{3t^2+4} = -\frac{12}{7} \rightarrow 3t^2 + 4 = -7t$

or $3t^2 + 7t + 4 = 0$

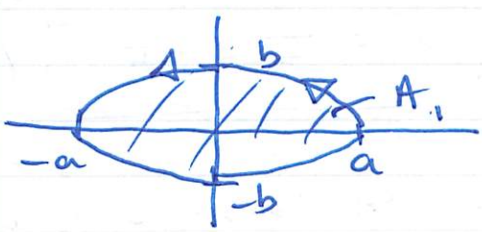
$(3t+4)(t+1) = 0$

$t = -1$; $t = -4/3$

$(x, y) = (-5, 6)$ or $(-\frac{208}{27}, \frac{32}{3})$

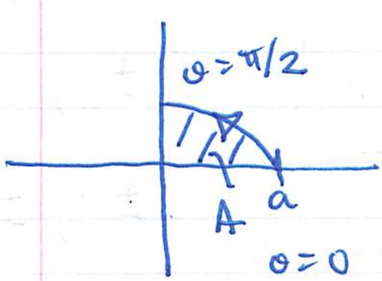
(31) use PE for an ellipse to find the area it encloses?

$x = a\cos\theta$; $y = b\sin\theta$; $0 \le \theta \le 2\pi$



$\cancel{A = 4A_0}$

Area $= 4A$



$A = \int_0 y\,dx = \int b\sin\theta\,[-a\sin\theta]\,d\theta$

$= -ab\int_{\pi/2} \sin^2\theta\,d\theta = ab\int_0^{\pi/2} \sin^2\theta\,d\theta$

$\frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$0 \le \theta \le \pi/2$

$$ab \int_0^{\pi/2} \left[\frac{1}{2} - \frac{1}{2}\cos(2\theta) \right] d\theta$$

$$= ab\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) \Big|_0^{\pi/2} \right]$$

$$ab\left[\frac{1}{2}\frac{\pi}{2} - \frac{1}{4}\cancel{\sin(\pi)}^{\,0} \right] - ab\left[\cancel{\frac{1}{2}0 - \frac{1}{4}\sin(0)}^{\,0} \right]$$

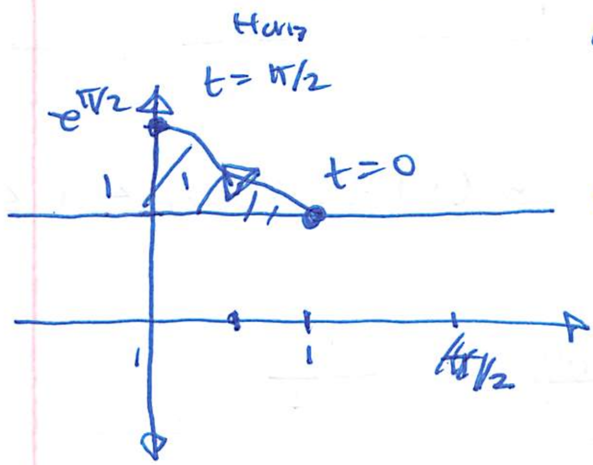
$$= \frac{1}{4}ab\pi \quad ; \quad 4A = \boxed{ab\pi}$$

(33) Find the area bounded by the curve

$x = \cos t$; $y = e^t$; $0 \le t \le \pi/2$

lines $y = 1$ and $x = 0$ (y-axis) $\quad t = 0 \quad (1, 1)$

$t = \pi/2 \quad (0, e^{\pi/2})$



$y = 1$; $A = \int_0^1 (y - 1)\, dx$

Substitute

$$A = \int_{\pi/2}^{0} (e^t - 1)\,[-\sin t]\, dt$$

$$= -\int_{\pi/2}^{0} (e^t - 1)\sin t\, dt = \int_0^{\pi/2} (e^t - 1)\sin t\, dt$$

$$= \cancel{4}\ \boxed{\frac{1}{2}\left(e^{\pi/2} - 1\right)}$$

Find the length of the curve

41 + 43

41 $\quad x = 1 + 3t^2 \; ; \; y = 4 + 2t^3 \; ; \; 0 \le t \le 1$

$$\frac{dx}{dt} = 6t \; ; \; \frac{dy}{dt} = 6t^2$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{(6t)^2 + (6t^2)^2} = \sqrt{36t^2 + 36t^4}$$

$$= \sqrt{36t^2(1+t^2)} = 6t\sqrt{1+t^2}$$

$$\int_0^1 6t\sqrt{1+t^2}\, dt = \boxed{2\left[2\sqrt{2} - 1\right]}$$

$u = 1 + t^2$

43 $\quad x = \frac{t}{1+t} \; ; \; y = \ln(1+t) \; ; \; 0 \le t \le 2$

$$\frac{dx}{dt} = \frac{1}{(1+t)^2} \; ; \; \frac{dy}{dt} = \frac{1}{1+t}$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{(1+t)^4} \; ; \; \left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+t)^2}$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{\frac{1}{(1+t)^4} + \frac{1}{(1+t)^2}}$$

$$= \cancel{\sqrt{\frac{1}{(1+t)^2}\Big[\frac{1}{(1+}}} = \sqrt{\frac{1 + (1+t)^2}{(1+t)^4}} =$$

$$\sqrt{\frac{1+(1+t)^2}{(1+t)^4}} = \frac{\sqrt{1+(1+t)^2}}{\cancel{t}\,(1+t)^2} - t^2+2t+2$$

$$\int_0^1 \frac{\sqrt{(1+t)^2+1}}{(1+t)^2}\,dt \quad ; \quad u = 1+t, \quad du = dt$$

$$\int_1^3 \frac{\sqrt{u^2+1}}{u^2}\,du$$ How can we integrate this ?

$$= -\frac{\sqrt{u^2+1}}{u} + \ln\left[u+\sqrt{u^2+1}\right]\Big|_1^3$$

$$= \boxed{-\frac{\sqrt{10}}{3} + \ln(3+\sqrt{10}) + \sqrt{2} - \ln(1+\sqrt{2})}$$

45) $x = e^t \cos t$; $y = e^t \sin t$; $0 \le t \le \pi$

$$(x')^2 + (y')^2 = \left[e^t(\cos t - \sin t)\right]^2 \cancel{+ [e^t \cos}$$

$$+ \left[e^t(\sin t + \cos t)\right]^2$$

$$= (e^t)^2\left[\cos^2 t - 2\cos t \sin t + \sin^2 t\right]$$

$$+ (e^t)^2\left[\sin^2 t + 2\sin t \cos t + \cos^2 t\right]$$

$$= e^{2t}\left[2\cos^2 t + 2\sin^2 t\right] = 2e^{2t}$$

$$\sqrt{} = \sqrt{2e^{2t}} = \sqrt{2}\,e^t$$

$$\int_0^{\pi} \sqrt{2}\, e^t \, dt = \sqrt{2}\, e^t \Big|_0^{\pi} = \boxed{\sqrt{2}\,(e^{\pi} - 1)}$$

Surface Area Rotate about x-axis

60 + 61

Surface area - Rotate about y axis.

66

60 $x = 3t - t^3$; $y = 3t^2$; $0 \le t \le 1$

$$(x')^2 + (y')^2 = (3 - 3t^2)^2 + (6t)^2 = 9(1 + 2t^2 + t^4)$$

$$= [3(1+t^2)]^2 \; ; \; \sqrt{\;} = |3(1+t^2)|$$

$$= 3(1 + t^2)$$

$$SA = \int 2\pi \underbrace{y}_{3t^2} \, ds \leftarrow 3(1 + t^2)$$

$$SA = \int 2\pi \cdot 3t^2 \cdot 3(1+t^2)\, dt = 18\pi \int_0^1 t^2 (1 + t^2)\, dt$$

$$= \boxed{\frac{48}{5}\pi}$$

(61) $x = a\cos^3\theta$; $y = a\sin^3\theta$; $0 \le \theta \le \frac{\pi}{2}$

$$(x')^2 + (y')^2 = [-3a\cos^2\theta\sin\theta]^2$$

$$+ [3a\sin^2\theta\cos\theta]^2$$

$$= 9a^2\sin^2\theta\cos^2\theta$$

$$SA = \int_0^{\pi/2} 2\pi a\sin^3\theta \cdot 3a\sin\theta\cos\theta\, d\theta$$

$$= 6\pi a^2 \int_0^{\pi/2} \sin^4\theta\cos\theta\, d\theta = \frac{6}{5}\pi a^2 \left[\sin^5\theta\right]_0^{\pi/2}$$

$$= \boxed{\frac{6}{5}\pi a^2}$$

(66) $x = e^t - t$; $y = 4e^{t/2}$; $0 \le t \le 1$

$$(x')^2 + (y')^2 = (e^t - 1)^2 + (2e^{t/2})^2$$

$$= e^{2t} + 2e^t + 1 = (e^t + 1)^2$$

$$SA = \int 2\pi x\, ds = \int_0^1 2\pi (e^t - t)(e^t + 1)\, dt$$

$$= \boxed{\pi(e^2 - 2e - 6)}$$