East Los Angeles College
Department of Mathematics
Math 262
Test 2

HH r

Solutions 140r

Determine the limit for the following.

1. $\lim_{x\to\infty} \frac{ln\sqrt{x}}{x^2}$ $\frac{\infty}{\infty}$

$\stackrel{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2x}}{2x}$

$= \lim_{x\to\infty} \frac{1}{4x^2}$

$= \boxed{0}$

sr

10 ✓

2. $\lim_{x\to\infty} xsin(\pi x)$ π/x

$\lim_{x\to\infty} x \sin(\frac{\pi}{x})$ $\infty \cdot 0$

$\underset{fix}{=} \lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x}$ $\frac{0}{0}$

$\stackrel{H}{=} \lim_{x\to\infty} \frac{\cos(\frac{\pi}{x}) \cdot (-\frac{\pi}{x^2})}{-\frac{1}{x^2}}$

$= \lim_{x\to\infty} \pi \cdot \cos(\frac{\pi}{x})$ $\boxed{\pi}$

sr

3. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$ $\quad \infty - \infty$

alg

$$= \lim_{x \to 0^+} \left(\frac{e^x - 1 - x}{x(e^x - 1)}\right) \quad \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 0^+} \frac{e^x - 1}{x e^x + e^x - 1} \quad \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 0^+} \frac{e^x}{x e^x + 2e^x} \quad \boxed{\frac{1}{2}}$$

sr

4. $\lim_{x \to \infty} (e^x + x)^{\frac{1}{x}}$ $\quad \infty^0$

let $y = (e^x + x)^{1/x}$

$\ln y = \ln (e^x + x)^{1/x}$

$\ln y = \frac{1}{x} \ln (e^x + x)$

as $x \to \infty$

$\ln y \to 0 \cdot \infty$

fix

$$\ln y = \frac{\ln(e^x + x)}{x}$$

as $x \to \infty$

$\ln y \to \frac{\infty}{\infty}$

H as $x \to \infty$

$\ln y \to \frac{e^x + 1}{e^x + x} \quad \frac{\infty}{\infty}$

H as $x \to \infty$

$\ln y \to \frac{e^x}{e^x + 1} = \frac{1}{1 + e^{-x}}$

$\to 1$

ie, $\ln y \to 1$

and $y \to e^1 = \boxed{e}$

sr

20 ✓

~~10~~ ✓

Integrate the following functions.

8. $\int tan^2(x)sec^4(x)dx$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$\sec^2(x) = \tan^2(x) + 1$

$\int u^2(u^2+1)\,du$

$\int (u^4+u^2)\,du$

$\frac{1}{5}u^5 + \frac{1}{3}u^3 + C$

$\boxed{\frac{1}{5}\tan^5(x) + \frac{1}{3}\tan^3(x) + C}$

$\frac{\cancel{8}}{10}$ ✓

9. $\int (2 - sinx)^2 dx$

$\int \left(4 - 4\sin(x) + \sin^2(x)\right)dx$

$\int \left(4 - 4\sin(x) + \frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx$

$\int \left(\frac{9}{2} - 4\sin(x) - \frac{1}{2}\cos(2x)\right)dx$

$\boxed{\frac{9}{2}x + 4\cos(x) - \frac{1}{4}\sin(2x) + C}$

$\frac{\cancel{8}}{10}$ ✓

$\cancel{18}$ ✓

20 ✓

10. $\int cos(2\pi x)cos(\pi x)dx$

Product to Sum

$$\int \left(\frac{1}{2} \cos(\pi x) + \frac{1}{2} \cos(3\pi x) \right) dx$$

u-sub

$u = \pi x$

$dx = \frac{du}{\pi}$

u-sub

$u = 3\pi x$

$dx = \frac{du}{3\pi}$

$$\boxed{\frac{1}{2\pi} \sin(\pi x) + \frac{1}{6\pi} \sin(3\pi x) + C}$$

8

10

11. $\int tan^3(x)\sec(x)\,dx$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$\tan^2(x) = \sec^2(x) - 1$

$$\int (u^2 - 1)\,du$$

$\frac{1}{3}u^3 - u + C$

$$\boxed{\frac{1}{3}\sec^3(x) - \sec(x) + C}$$

8

10

18

20

Integrate the following functions.

12. $\int \sqrt{1-4x^2}dx$ (u-sub)

$u = 2x \; ; \quad dx = \frac{du}{2}$

$\frac{1}{2}\int \sqrt{1-u^2}\, du$ (Trig sub)

$u = \sin\theta \; ; \quad du = \cos\theta\, d\theta$

$\sqrt{1-u^2} = \cos\theta$

$\frac{1}{2}\int \cos^2\theta\, d\theta$

$\frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$

$\sin\theta = u \; ; \quad \sin(2\theta) = 2\sin\theta\cos\theta$

$\frac{1}{4}\sin^{-1}(u) + \frac{1}{4}u\sqrt{1-u^2} + C$

$u = 2x$

$$\boxed{\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C}$$

10✓

13. $\int \frac{x}{\sqrt{x^2+x+1}}dx$ $\quad x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$

$u = x + \frac{1}{2} \; ; \quad du = dx$

$\int \frac{u-\frac{1}{2}}{\sqrt{u^2+\frac{3}{4}}}\, du = \int \frac{2u-1}{2\sqrt{u^2+\frac{3}{4}}}\, du$

Trig sub

$u = \frac{\sqrt{3}}{2}\tan\theta$

$du = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$

$\sqrt{u^2+\frac{3}{4}} = \frac{\sqrt{3}}{2}\sec\theta$

$\frac{1}{2}\int \frac{\sqrt{3}\tan\theta - 1}{\frac{\sqrt{3}}{2}\sec\theta}\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$

$\frac{1}{2}\int (\sqrt{3}\tan\theta - 1)\sec\theta\, d\theta$

$\frac{1}{2}\int (\sqrt{3}\sec\theta\tan\theta - \sec\theta)\, d\theta$

$\frac{\sqrt{3}}{2}\int \sec\theta\tan\theta\, d\theta - \frac{1}{2}\int \sec\theta\, d\theta$

$\frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|$

10✓

But, $u = \frac{\sqrt{3}}{2}\tan\theta \; , \quad \tan\theta = \frac{2u}{\sqrt{3}}$

$\sec\theta = \frac{\sqrt{3+4u^2}}{\sqrt{3}} \; ; \quad u = x + \frac{1}{2}$

20✓

$\frac{\sqrt{3+4u^2}}{2} - \frac{1}{2}\ln\left|\frac{\sqrt{3+4u^2}}{\sqrt{3}} + \frac{2u}{\sqrt{3}}\right|$

$\frac{\sqrt{3+4(x+\frac{1}{2})^2}}{2} - \frac{1}{2}\ln\left|\frac{\sqrt{3+4(x+\frac{1}{2})^2}}{\sqrt{3}} + \frac{2(x+\frac{1}{2})}{\sqrt{3}}\right| + C$

$$\boxed{\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\frac{2\sqrt{x^2+x+1} + 2x+1}{\sqrt{3}}\right| + C}$$

14. Determine the average value of the function $f(x) = \frac{\sqrt{x^2-1}}{x}$ over the interval $[1,4]$

$$f_{avg} = \frac{1}{3}\int_1^4 \frac{\sqrt{x^2-1}}{x}\,dx \qquad \text{use trig Sub}$$

$$x = \sec\theta \;; \quad dx = \sec\theta\tan\theta\,d\theta$$

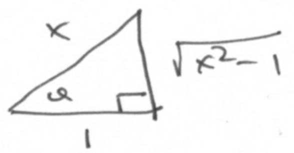
$$\sqrt{x^2-1} = \tan\theta$$

$$\frac{1}{3}\int_{x=1}^{x=4} \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta$$

$$\frac{1}{3}\int \tan^2\theta\,d\theta = \frac{1}{3}\int (\sec^2\theta - 1)\,d\theta$$

$$\frac{1}{3}\Big[\tan\theta - \theta\Big]_{x=1}^{x=4}$$

But, $\sec\theta = \frac{x}{1}$; $\cos\theta = \frac{1}{x}$;



; $\tan\theta = \frac{\sqrt{x^2-1}}{1}$; $\theta = \cos^{-1}\left(\frac{1}{x}\right)$

$$\frac{1}{3}\left[\frac{\sqrt{x^2-1}}{1} - \cos^{-1}\left(\frac{1}{x}\right)\Bigg|_{x=1}^{x=4}\right]$$

10 ✓

$$\boxed{\frac{1}{3}\left[\sqrt{15} - \cos^{-1}\left(\frac{1}{4}\right)\right]}$$

15. Find the area of the region under the given curve $y = \frac{1}{x^3+x}$ from $1 \le x \le 2$

$$\int_1^2 \frac{1}{x^3+x}\,dx$$

$$\int_1^2 \frac{1}{x(x^2+1)}\,dx \qquad \text{pfd}$$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad ; \quad A=1 \quad B=-1 \quad C=0$$

$$\int_1^2 \frac{1}{x} - \frac{x}{x^2+1}\,dx$$

$$\int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx$$

u-Sub

$$\ln|x| - \frac{1}{2}\ln\left(\sqrt{x^2+1}\right)\Big|_{x=1}^{x=2}$$

$$\ln(2) - \frac{1}{2}\ln\sqrt{5} + \frac{1}{2}\ln\sqrt{2}$$

$$\ln(2) + \frac{1}{2}\ln\sqrt{2} - \frac{1}{2}\ln\sqrt{5}$$

$$\ln(2) + \frac{1}{2}\ln\left(\frac{\sqrt{2}}{\sqrt{5}}\right)$$

10r

$$\ln(2) + \frac{1}{4}\ln\left(\frac{2}{5}\right)$$

math 262 Test 2

(1) $\lim_{x \to \infty} \frac{\ln \sqrt{x}}{x^2}$ $\frac{\infty}{\infty}$

$\underset{H}{=} \lim_{x \to \infty} \frac{\frac{d}{dx}(\ln \sqrt{x})}{\frac{d}{dx}(x^2)}$

$= \lim_{x \to \infty} \frac{\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{2x}$ $\frac{\frac{1}{2x}}{2x}$ $\frac{1}{2x} \div 2x$ $\frac{1}{4x^2}$

$\lim_{x \to \infty} \frac{1}{4x^2} = \boxed{0}$

(2) $\lim_{x \to \infty} x \cdot \sin\left(\frac{\pi}{x}\right)$ $\infty \cdot 0$

$\lim_{x \to \infty} \frac{\sin\left(\frac{\pi}{x}\right)}{1/x}$ $\frac{0}{0}$

$\underset{H}{=} \lim_{x \to \infty} \frac{\frac{d}{dx}\left(\sin\left(\frac{\pi}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$ $\cos\left(\frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)$ $-\frac{1}{x^2}$

$= \lim_{x \to \infty} \frac{-\frac{\pi}{x^2} \cos\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$

$$= \lim_{x \to \infty} -\pi \cos\left(\frac{\pi}{x}\right)$$

$$= -\pi \lim_{x \to \infty} \cos\left(\frac{\pi}{x}\right) \xrightarrow{} 1 \qquad = \boxed{-\pi}$$

(3) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$ $\quad \infty - \infty$

use algebra

$\boxed{\text{algebra}}$ $\quad \frac{1}{x} - \frac{1}{e^x - 1}$

$$\lim_{x \to 0^+} \left(\frac{e^x - 1 - x}{x(e^x - 1)} \right) = \frac{e^0 - 1 - 0}{0\,(e^0 - 1)} \qquad \frac{0}{0}$$

$$\underset{H}{=} \lim_{x \to 0^+} \frac{\frac{d}{dx}(e^x - 1 - x)}{\frac{d}{dx}(xe^x - x)}$$

$$= \lim_{x \to 0^+} \frac{e^x - 1}{xe^x + e^x - 1} \quad \frac{e^0 - 1}{0e^0 + e^0 - 1} \qquad \frac{0}{0}$$

$$\underset{H}{=} \lim_{x \to 0^+} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(xe^x + e^x - 1)}$$

$$= \lim_{x \to 0^+} \frac{e^x}{xe^x + e^x + e^x}$$

$$= \lim_{x \to 0^+} \frac{e^x}{xe^x + 2e^x} \quad \frac{e^0}{0e^0 + 2e^0} \quad \boxed{\frac{1}{2}}$$

(4) $\lim_{x \to \infty} (e^x + x)^{1/x}$ $\quad \infty^0$

let $y = (e^x + x)^{1/x}$; $\ln y = \ln\left[(e^x + x)^{1/x}\right]$

$$\ln y = \frac{1}{x} \ln (e^x + x)$$

$$\lim_{x \to \infty} \frac{1}{x} \ln(e^x + x) \quad 0 \cdot \infty$$

fix

But, $\lim_{x \to \infty} \frac{\ln (e^x + x)}{x}$ $\quad \frac{\infty}{\infty}$

$$\underset{H}{=} \lim_{x \to \infty} \frac{\frac{d}{dx}\left[\ln (e^x + x)\right]}{\frac{d}{dx}(x)}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{e^x + x} \cdot (e^x + 1)}{1}$$

$$\lim_{x\to\infty} \frac{e^x+1}{e^x+x} \quad \frac{\infty}{\infty}$$

$$\overset{=}{H} \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x+1)}{\frac{d}{dx}(e^x+x)}$$

$$= \lim_{x\to\infty} \frac{e^x}{e^x+1} \quad \frac{\infty}{\infty}$$

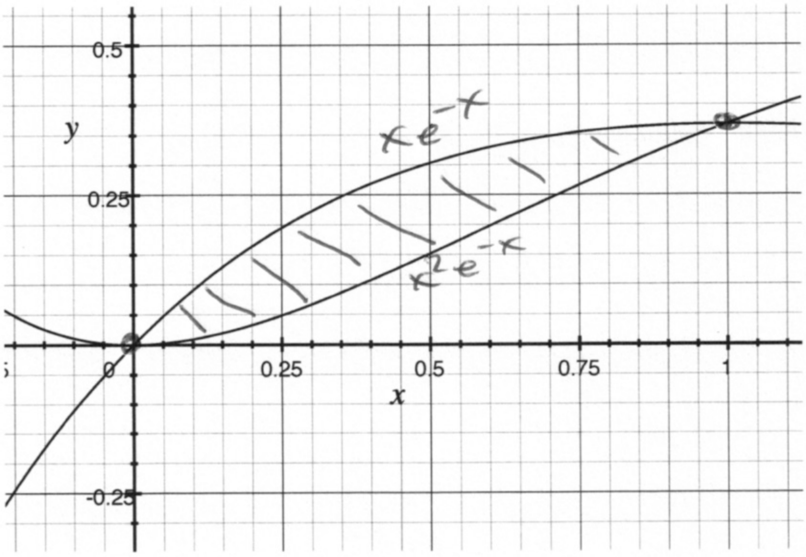
But, $\dfrac{e^x / e^x}{\frac{e^x+1}{e^x} / e^x} \qquad \dfrac{1}{1+e^{-x}}$

$$\lim_{x\to\infty} \frac{1}{1+e^{-x}} = \frac{1}{1+0} \quad 1$$

ie, as $x\to\infty$; $\ln y \to 1$

or $y \to e^1 = \boxed{e}$

(5)



pts of intersection

$x^2e^{-x} = xe^{-x}$

$x^2 = x$;

$x^2 - x = 0$

$x(x-1) = 0$

$x=0$ | $x-1=0$
$+1 \quad +1$
$x=1$

note which curve is the top curve?

let $x = \frac{1}{2}$

Bottom

$y = x^2e^{-x}$

$y = (\frac{1}{2})^2 e^{-1/2}$

$y = \frac{1}{4}e^{-0.5}$

Bottom

Top

$y = x e^{-x}$

$y = \frac{1}{2}e^{-1/2}$

$y = \frac{1}{2}e^{-0.5}$

TOP

$$A = \int_0^1 (\text{Top} - \text{Bottom})\,dx = \int_0^1 (xe^{-x} - x^2e^{-x})\,dx$$

Better to integrate separate

$$= \int_0^1 xe^{-x}(1-x)\,dx$$

ie, $A = \int_0^1 xe^{-x}\,dx - \int_0^1 x^2e^{-x}\,dx$

use IBP for both integrals

A_1 $\qquad$ A_2

$$A_2 = \int_0^1 \underset{u}{x^2} \underbrace{e^{-x}\,dx}_{dv}$$

$u = x^2$; $dv = e^{-x}\,dx$

$du = 2x\,dx$; $v = \int e^{-x}\,dx = -e^{-x}$

$$uv - \int v\,du = -x^2 e^{-x} - \int (-e^{-x})\,2x\,dx$$

$$= -x^2 e^{-x} + 2 \underbrace{\int x e^{-x}\,dx}_{IBP}$$

$u = x$ $\quad dv = e^{-x}\,dx$

$du = dx$ $\quad v = \int e^{-x}\,dx$

$uv - \int v\,du$ $\quad v = -e^{-x}$

$$-xe^{-x} - \int -e^{-x}\,dx$$

$$-xe^{-x} + \int e^{-x}\,dx$$

$$-xe^{-x} - e^{-x}$$

ie, $A_2 = \int_0^1 x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\left[-xe^{-x} - e^{-x}\right]$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}$$

$$= -e^{-x}(x^2 + 2x + 2)\Big|_0^1$$

$$= -e^{-1}(1^2 + 2 + 2) - \left[-e^0\,(0^2 + 2\cdot 0 + 2)\right]$$

$$A_2 = -\frac{1}{e} \cdot 5 - [-1\ (2)]$$

$$= -\frac{5}{e} - (-2)$$

$$\boxed{A_2 = 2 - \frac{5}{e}}$$

$$A_1 = \int_0^1 x e^{-x}\, dx$$

IBP

$$A_1 = -x e^{-x} - e^{-x} \Big|_0^1$$

$$= -1 e^{-1} - e^{-1} - [-0 e^0 - e^0]$$

$$= -\frac{1}{e} - \frac{1}{e} + 1$$

$$\boxed{A_1 = 1 - \frac{2}{e}}$$

$$A = A_1 + A_2$$

$$A = 1 - \frac{2}{e} + 2 + \frac{5}{e}$$

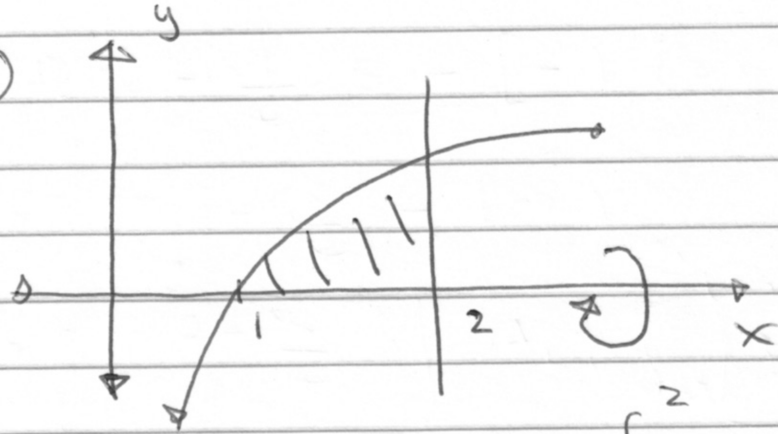
$$A = \cancel{3} \cancel{- 7/e}$$

$$\boxed{\cancel{A = \frac{3e - 7}{e}}}$$

$$A = -1 + \frac{3}{e}$$

$$\boxed{A = \frac{3}{e} - 1}$$

(6)



Disk Method ; $V = \int_1^2 \pi r^2 \, dx$

$r = \ln(x)$

$V = \int_1^2 \pi [\ln(x)]^2 \, dx = \pi \int_1^2 \ln^2(x) \, dx$

what is $\int_1^2 \ln^2(x) \, dx$?

$\int_1^2 \underset{u}{\ln(x)} \, \underbrace{\ln(x) \, dx}_{dv}$ use IBP

$u = \ln(x)$ $dv = \ln(x) \, dx$

$du = \frac{1}{x} \, dx$ $v = \int \ln(x) \, dx$

$v = x \ln(x) - x$

$uv - \int v \, du$

$\ln(x) [x \ln(x) - x] - \int [x \ln(x) - x] \frac{1}{x} \, dx$

$x \ln^2 x - x \ln(x) - \int [\ln(x) - 1] \, dx$

$$x\ln^2(x) - x\ln(x) - \underbrace{\int \ln(x)\,dx}_{x\ln(x) - x} + \underbrace{\int dx}_{x}$$

$$x\ln^2(x) - x\ln(x) - \left(x\ln(x) - x\right) + x$$

$$x\ln^2(x) - x\ln(x) - x\ln(x) + x + x$$

$$x\ln^2(x) - 2x\ln(x) + 2x \Big|_{x=1}^{x=2}$$

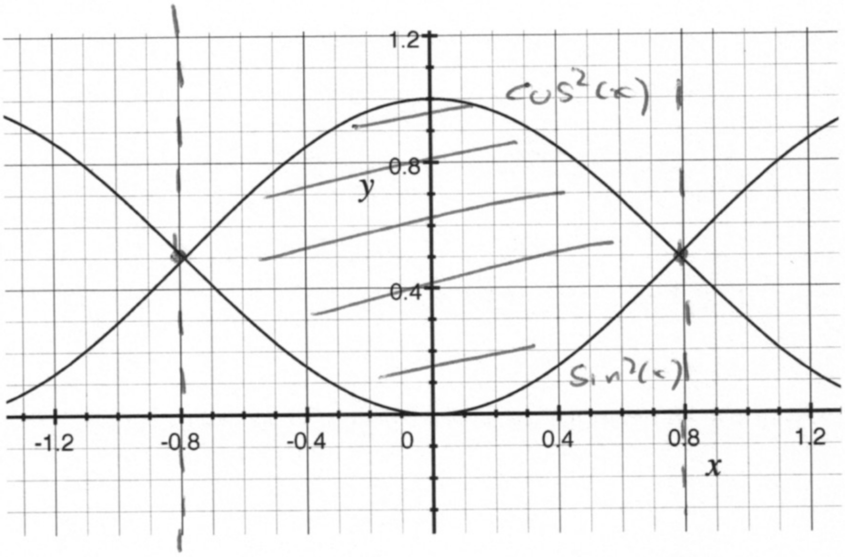
$$2\ln^2(2) - 4\ln(2) + 4 - \left(\overset{0}{\cancel{\ln^2(1)}} - \overset{0}{\cancel{2\ln(1)}} + 2\right)$$

$$2\ln^2(2) - 4\ln(2) + 4 - 2$$

$$2\ln^2(2) - 4\ln(2) + 2$$

ie, $\boxed{V = \pi\left[2\ln^2(2) - 4\ln(2) + 2\right]}$

7



$-\frac{\pi}{4}$ $\frac{\pi}{4}$

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\text{top} - \text{Bottom})\,dx = \int_{-\frac{\pi}{4}}^{\pi/4} (\cos^2(x) - \sin^2(x))\,dx$$

$\frac{1}{2}[1+\cos(2x)]$ $\frac{1}{2}[1-\cos(2x)]$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \frac{1}{2}[1+\cos(2x)] - \frac{1}{2}[1-\cos(2x)]\,dx$$

algebra

$\frac{1}{2} + \frac{1}{2}\cos(2x) - \frac{1}{2} + \frac{1}{2}\cos(2x)$

$\cos(2x)$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \cos(2x)\,dx = \frac{1}{2}\sin(2x)\Big|_{-\pi/4}^{\pi/4}$$

u-sub

$u = 2x$

$$= \frac{1}{2}\sin\left(2\frac{\pi}{4}\right) - \left[\frac{1}{2}\sin\left(2\left(-\frac{\pi}{4}\right)\right)\right]$$

$\frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$

$\frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)$

$\sin\left(\frac{\pi}{2}\right)$ $\boxed{1}$

(8) $\int \tan^2(x) \sec^4(x)\, dx$

$$\Downarrow$$

$$\sec^2(x) \sec^2(x)$$

$$\Downarrow$$

$$\tan^2 x + 1$$

$$\int \tan^2(x) \left(\tan^2(x) + 1\right) \underbrace{\sec^2(x)\, dx}_{du}$$

$u = \tan(x)$

$du = \sec^2(x)\, dx$

$$\int u^2 (u^2+1)\, du = \int (u^4 + u^2)\, du$$

$$\frac{1}{5} u^5 + \frac{1}{3} u^3 + C$$

$$\boxed{\frac{1}{5} \tan^5(x) + \frac{1}{3} \tan^3(x) + C}$$

(9) $\int (2 - \sin x)^2\, dx$

algebra

$(2 - \sin x)(2 - \sin x) = 4 - 2\sin x - 2\sin x + \sin^2 x$

$= 4 - 4\sin(x) + \sin^2(x)$

$$\Downarrow$$

$$\frac{1}{2}\left[1 - \cos(2x)\right]$$

$$4 - 4\sin(x) + \frac{1}{2}\left[1 - \cos(2x)\right]$$

$$4 - 4\sin(x) + \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\frac{9}{2} - 4\sin(x) - \frac{1}{2}\cos(2x)$$

$$\int\left[\frac{9}{2} - 4\sin(x) - \frac{1}{2}\cos(2x)\right]dx$$

$$\frac{9}{2}\int dx - 4\int \sin(x)\,dx - \frac{1}{2}\int \cos(2x)\,dx$$

u-sub

$u = 2x$

$du = 2\,dx$

$$\boxed{\frac{9}{2}x + 4\cos(x) - \frac{1}{4}\sin(2x) + C}$$

(10) $\int \cos(2\pi x)\cos(\pi x)\,dx$

product to sum

$$\frac{1}{2}\left[\cos(2\pi x - \pi x) + \cos(2\pi x + \pi x)\right]$$

$$\frac{1}{2}\left[\cos(\pi x) + \cos(3\pi x)\right]$$

$$\frac{1}{2}\cos(\pi x) + \frac{1}{2}\cos(3\pi x)$$

$$\int\left[\frac{1}{2}\cos(\pi x) + \frac{1}{2}\cos(3\pi x)\right]dx$$

$$\frac{1}{2}\int \cos(\pi x)\,dx + \frac{1}{2}\int \cos(3\pi x)\,dx$$

u-sub u-sub

$$\frac{1}{2}\int \cos(\pi x)\,dx + \frac{1}{2}\int \cos(3\pi x)\,dx$$

$u = \pi x$ $\quad$ $u = 3\pi x$

$du = \pi\,dx$ $\quad$ $du = 3\pi\,dx$

$dx = \frac{du}{\pi}$ $\quad$ $dx = \frac{du}{3\pi}$

$$\frac{1}{2}\frac{\sin(\pi x)}{\pi} + \frac{1}{2}\frac{\sin(3\pi x)}{3\pi} + C$$

$$\boxed{\frac{1}{2\pi}\sin(\pi x) + \frac{1}{6\pi}\sin(3\pi x) + C}$$

(11) $\int \tan^3(x)\sec(x)\,dx$

$\tan^2(x)\,\sec(x)\,\tan(x)$

$\sec^2(x) - 1$

$$\int (\sec^2(x) - 1)\sec(x)\tan(x)\,dx$$

$u = \sec(x)$; $\quad \frac{du}{dx} = \sec(x)\tan(x)$

$du = \sec(x)\tan(x)\,dx$

$$\int (u^2 - 1)\,du$$

$$\frac{1}{3}u^3 - u + C$$

$$\boxed{\frac{1}{3}\sec^3(x) - \sec(x) + C}$$

(12) $\int \sqrt{1-4x^2}\ dx$

$= \int \sqrt{1-(2x)^2}\ dx$ let $u = 2x$

$du = 2\,dx$

$dx = \frac{du}{2}$

$= \int \sqrt{1-u^2}\ \frac{du}{2}$

$= \frac{1}{2}\int \sqrt{1-u^2}\ du$ use Trig sub

a^2 ; $a^2 = 1$ or $a = 1$

$u = \sin\theta$; $du = \cos\theta\ d\theta$;

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta}$

$= |\cos\theta| = \cos\theta$

$\frac{1}{2}\int \cos\theta \cdot \cos\theta\ d\theta = \frac{1}{2}\int \cos^2\theta\ d\varphi$

$\frac{1}{2}\left[1+\cos(2\theta)\right]$

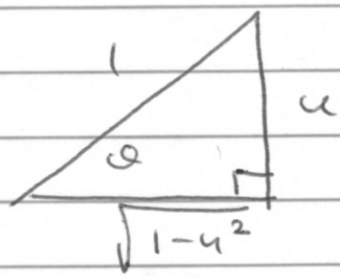
$= \frac{1}{2}\int \frac{1}{2}\left[1+\cos(2\theta)\right] d\theta$

$$\frac{1}{4}\int [1 + \cos(2\theta)]\, d\theta$$

$$\frac{1}{4}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

$$\frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$$

But, $\sin\theta = \frac{u}{1}$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$



ie, $\sin(2\theta) = 2u \cdot \frac{\sqrt{1-u^2}}{1}$

$= 2u\sqrt{1-u^2}$

$\sin\theta = u \rightarrow \theta = \sin^{-1}(u)$

$$\frac{1}{4}\sin^{-1}(u) + \frac{1}{8}\cdot 2u\sqrt{1-u^2} + C$$

$$\frac{1}{4}\sin^{-1}(u) + \frac{1}{4}u\sqrt{1-u^2} + C$$

But, $u = 2x$

$$\frac{1}{4}\sin^{-1}(2x) + \frac{1}{4}2x\sqrt{1-(2x)^2} + C$$

$$\boxed{\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C}$$

(13) $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

complete the square ; $\frac{b}{2} = \frac{1}{2}$; $\left(\frac{b}{2}\right)^2 = \frac{1}{4}$

$$x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1$$

$$= \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

let $u = x+\frac{1}{2}$; $x = u-\frac{1}{2}$

$du = dx$

$$\int \frac{u-\frac{1}{2}}{\sqrt{u^2+\frac{3}{4}}}\,du = \int \frac{2\left(u-\frac{1}{2}\right)}{2\sqrt{u^2+\frac{3}{4}}}\,du$$

$$= \int \frac{2u-1}{2\sqrt{u^2+\frac{3}{4}}}\,du = \frac{1}{2}\int \frac{2u-1}{\sqrt{u^2+\frac{3}{4}}}\,du$$

Trig Sub

$u = \frac{\sqrt{3}}{2}\tan\theta$

$du = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$\sqrt{u^2+\frac{3}{4}} = \frac{\sqrt{3}}{2}\sec\theta$

use trig sub

$a^2 = \frac{3}{4}$; $a = \frac{\sqrt{3}}{2}$

$\cancel{u = \frac{3}{4}\tan\theta}$

$\cancel{du = \frac{3}{4}\sec^2\theta\,d\theta}$

$\cancel{\sqrt{u^2+\frac{3}{4}} = \frac{3}{4}}$

$$\frac{1}{2}\int \frac{\left(2\frac{\sqrt{3}}{2}\tan\theta - 1\right)}{\cancel{\frac{\sqrt{3}}{2}}\sec\cancel{\theta}} \cdot \cancel{\frac{\sqrt{3}}{2}}\sec^2\theta \, d\theta$$

$$\frac{1}{2}\int (\sqrt{3}\tan\theta - 1)\sec\theta \, d\theta$$

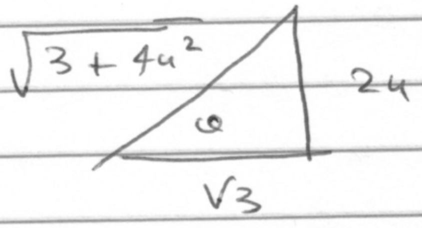
$$\frac{1}{2}\int (\sqrt{3}\sec\theta\tan\theta - \sec\theta)\, d\theta$$

$$\frac{\sqrt{3}}{2}\int \sec\theta\tan\theta \, d\theta - \frac{1}{2}\int \sec\theta \, d\theta$$

$$\frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

But, $u = \frac{\sqrt{3}}{2}\tan\theta$

ie, $\tan\theta = \frac{2u}{\sqrt{3}}$



so, $\sec\theta = \frac{\sqrt{3+4u^2}}{\sqrt{3}}$

$$\frac{\sqrt{3}}{2}\frac{\sqrt{3+4u^2}}{\sqrt{3}} - \frac{1}{2}\ln\left|\frac{\sqrt{3+4u^2}}{\sqrt{3}} + \frac{2u}{\sqrt{3}}\right|$$

$$\frac{1}{2}\sqrt{3+4u^2} - \frac{1}{2}\ln\left|\frac{\sqrt{3+4u^2}}{\sqrt{3}} + \frac{2u}{\sqrt{3}}\right|$$

But $u = x + \frac{1}{2}$

$$\frac{1}{2}\sqrt{3+4(x+\tfrac{1}{2})^2} - \frac{1}{2}\ln\left|\sqrt{3+4(x+\tfrac{1}{2})^2} + \frac{2(x+}{\sqrt{3}}\right.$$

ie,

$$\frac{1}{2}\sqrt{3+4(x+\tfrac{1}{2})^2}$$

$$- \frac{1}{2}\ln\left|\frac{\sqrt{3+4(x+\tfrac{1}{2})^2}}{\sqrt{3}} + \frac{2(x+\tfrac{1}{2})}{\sqrt{3}}\right|$$

$$+ C$$

note : $\sqrt{3+4(x+\tfrac{1}{2})^2} = \sqrt{3+4(x^2+x+\tfrac{1}{4})}$

$$= \sqrt{3+4x^2+4x+1}$$

$$\sqrt{4x^2+4x+4} = \sqrt{4}\sqrt{x^2+x+1}$$

$$2\sqrt{x^2+x+1}$$

$$\frac{2\sqrt{x^2+x+1}}{2} - \frac{1}{2}\ln\left|\frac{2\sqrt{x^2+x+1}}{\sqrt{3}} + \frac{2x+1}{\sqrt{3}}\right| + C$$

$$\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\frac{2\sqrt{x^2+x+1}}{\sqrt{3}} + \frac{2x+1}{\sqrt{3}}\right| + C$$

$$\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\frac{2\sqrt{x^2+x+1} + 2x+1}{\sqrt{3}}\right| + C$$

$$\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|2\sqrt{x^2+x+1} + 2x+1\right| - \frac{1}{2}\ln(3) + C$$

$$\boxed{\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|2\sqrt{x^2+x+1} + 2x+1\right| + C}$$

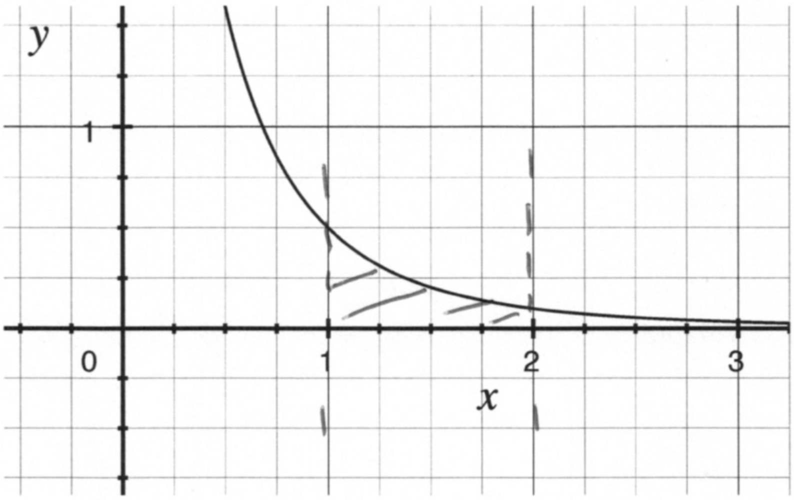
(14) $\frac{1}{3}\left[\sqrt{x^2-1} - \cos^{-1}\left(\frac{1}{x}\right)\Big|_{x=1}^{x=4}\right]$

$= \frac{1}{3}\sqrt{4^2-1} - \cos^{-1}\left(\frac{1}{4}\right)$

$- \frac{1}{3}\sqrt{1^2-1} + \cos^{-1}(1)$

$= \frac{1}{3}\sqrt{15} - \cos^{-1}\left(\frac{1}{4}\right) + \cancel{\cos^{-1}(1)}$ 0

(15) $y = \dfrac{1}{x^3+x}$ from $1 \le x \le 2$



$$\int_1^2 \frac{1}{x^3+x}\,dx = \int_1^2 \frac{1}{x(x^2+1)}\,dx$$ use partial fraction decomposition

ie, $$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + x(Bx+C)}{x(x^2+1)}$$

ie, $1 = Ax^2 + A + Bx^2 + Cx$

$1 = (A+B)x^2 + Cx + A$

ie, $A+B=0$ ie, $B=-1$

$C=0$

$A=1$

$$\frac{1}{3}\left[\tan(4) - 4\right] - \frac{1}{3}\left[\tan(1) - 1\right]$$

$$\frac{1}{3}\tan(4) - \frac{4}{3} - \frac{1}{3}\tan(1) + \frac{1}{3}$$

$$\boxed{\frac{1}{3}\tan(4) - \frac{1}{3}\tan(1) - 1}$$

(15) continuation

$$\int_1^2 \frac{1}{x^3+x}\,dx = \int_1^2 \left(\frac{1}{x} + \frac{-x}{x^2+1}\right) dx$$

$$= \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx$$

u-sub

$\ln|x|$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\frac{1}{2}\int \frac{du}{u} = \ln|u|$$

$$\frac{1}{2}\ln|x^2+1|$$

$$\ln\sqrt{x^2+1}$$

$$A = \ln|x| - \ln\sqrt{x^2+1} \Bigg|_1^2$$

$$A = \ln(2) - \ln\sqrt{5}$$

$$- \cancel{\ln(1)}^{0} + \ln\sqrt{2}$$

$$\boxed{A = \ln(2) - \ln(\sqrt{5}) + \ln(\sqrt{2})}$$