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East Los Angeles College
Department of Mathematics
Math 262
Test 3

Evaluate the following improper integrals.

1. $\int_1^{\infty} \frac{tan^{-1}(x)}{x} dx$

2. $\int_0^1 \frac{t^2+1}{t^2-1} dt$ $= \lim_{s \to 1^-} \int_0^s \frac{t^2+1}{t^2-1} dt$; $I = \int 1 + \frac{2}{t^2-1} dt$

$= \int dt + 2\int \frac{1}{t^2-1} dt$

$I = s + \ln\left|\frac{s+1}{s-1}\right|$

as $s \to 1^-$; $I \to \infty$

Diverges

4

3. Determine the arc length for the curve $y = 2ln\left(sin\left(\frac{1}{2}x\right)\right)$ over $\frac{\pi}{3} \le x \le \pi$.

$y' = \cot\left(\frac{1}{2}x\right)$; $\sqrt{1+(y')^2} = \csc\left(\frac{1}{2}x\right)$ ✓

✓

$s = \int_{\frac{\pi}{3}}^{\pi} \csc\left(\frac{1}{2}x\right) dx = 2\ln\left|\csc\left(\frac{1}{2}x\right) - \cot\left(\frac{1}{2}x\right)\right| \Big|_{\frac{\pi}{3}}^{\pi}$

u-Sub

✓ ✓

$= \boxed{-2\ln\left(2-\sqrt{3}\right)}$

✓

4. Determine the surface area of the solid by rotating the curve $y = x^2$ over $0 \le x \le 1$ about the y-axis.

$SA = \int 2\pi\, r\, ds$ ✓ $= \int_0^1 2\pi x \sqrt{1+4x^2}\, dx$

$r = x$, $ds = \sqrt{1+4x^2}$

u = Sub

✓ ✓

$= \boxed{\frac{\pi}{6}\left[5\sqrt{5}-1\right]}$ ✓

✓

8

~~10~~ ✓

5. Approximate (Thousandths) the surface area of the solid by rotating the curve $y = x^2$ over $0 \le x \le 1$ about the x-axis using Simpson's Rule with n=6

$$SA = \int 2\pi \underset{y}{r}\, \underset{\sqrt{1+(y')^2}}{ds} = \int 2\pi x^2 \sqrt{1+4x^2}\, dx$$ ✓

$\Delta x = \frac{1}{6}$ ✓ ; $f(x) = x^2\sqrt{1+4x^2}$

$$S_6 = \frac{\pi}{9}\Big[f(0) + 4f(\tfrac{1}{6}) + 2f(\tfrac{1}{3}) + 4f(\tfrac{1}{2}) + 2f(\tfrac{2}{3}) + 4f(\tfrac{5}{6}) + f(1)\Big]$$ ✓ ✓

$$\boxed{S_6 \approx 3.810}$$ ✓

6. Determine the length of the curve.

$$x = \frac{t}{1+t}$$
$$y = ln(1+t)$$
$$0 \le t \le 2$$

$$s = \int_0^2 \frac{\sqrt{(t+1)^2+1}}{(t+1)^2}\, dt$$ ✓ ; $u = t+1$, $du = dt$ ✓

$$= \int_1^3 \frac{\sqrt{u^2+1}}{u}\, du = -\frac{\sqrt{u^2+1}}{u} + \ln\left(u + \sqrt{u^2+1}\right)\Big|_{u=1}^{u=3}$$ ✓ ✓

pfd

$$= \boxed{\ln\left(\frac{3+\sqrt{10}}{1+\sqrt{2}}\right) + \sqrt{2} - \frac{\sqrt{10}}{3}}$$ ✓

10 ✓

7. Determine the points where the tangent line is horizontal and vertical.

$$x = 10 - t^2$$
$$y = t^3 - 12t$$

HT $\frac{dy}{dt} = 0$; $3t^2 - 12 = 0$; $t = \pm 2$ ✓

$t = 2$ $\boxed{(6, -16)}$ ✓ $t = -2$ $\boxed{(6, 16)}$ ✓

VT $\frac{dx}{dt} = 0$; $-2t = 0$; $t = 0$ ✓

$t = 0$ $\boxed{(10, 0)}$ ✓ ✓

8 ✓

(2) $\int_0^1 \frac{t^2+1}{t^2-1}\,dt = \int_0^1 1 + \frac{2}{t^2-1}\,dt$

$= \int_0^1 dt + 2\left[\int_0^1 \frac{1}{t^2-1}\,dt\right]$ Improper Integrals!

$\int_0^1 \frac{1}{t^2-1}\,dt = \lim_{s\to 1^-} \int_0^s \frac{1}{t^2-1}\,dt$

$(t+1)(t-1)$

$\frac{1}{t^2-1} = \frac{A}{t+1} + \frac{B}{t-1}$; $A = \frac{1}{2}$, $B = -\frac{1}{2}$

$= \frac{1/2}{t+1} - \frac{1/2}{t-1}$

$I = \int_0^s \frac{1}{t^2-1}\,dt = \frac{1}{2}\int_0^s \frac{1}{t+1} - \frac{1}{t-1}\,dt$

$= \frac{1}{2}\left[\ln|t+1| - \ln|t-1|\right] = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|\Big|_0^s$

~~as $s\to 1^-$; $|I| \to 0$~~

$\frac{1}{2}\ln\left|\frac{s+1}{s-1}\right| - \cancel{\frac{1}{2}\ln\left|\frac{1}{-1}\right|} = \frac{1}{2}\ln\left|\frac{s+1}{s-1}\right|$

as $s \to 1^-$; $I = \frac{1}{2}\ln\left|\frac{s+1}{s-1}\right| \to ?$

$$\begin{array}{r} 1 \\ s-1 \,\overline{)\, s+1} \\ +\; s+1 \\ \hline 2 \end{array}$$

$1 + \frac{2}{s-1}$

so, as $s \to 1^-$; $1 + \frac{2}{s-1} \to -\infty$

so $I = \frac{1}{2} \ln \left| 1 + \frac{2}{s-1} \right| \to \infty$

ie, Diverges

(3)

$$y = 2 \ln \left[\sin \left(\tfrac{1}{2}x \right) \right] \quad ; \quad \frac{\pi}{3} \le x \le \pi$$

$$y' = 2 \frac{d}{dx} \left[\ln \left(\sin \left(\tfrac{1}{2}x \right) \right) \right]$$

$$= 2 \frac{1}{\sin(\frac{1}{2}x)} \frac{d}{dx} \left[\cancel{\sin \left(\tfrac{1}{2}x \right)} \right] \quad \cos\left(\tfrac{1}{2}x\right) \cdot \frac{1}{2}$$

$$= \frac{\cos \left(\frac{1}{2}x \right)}{\sin \left(\frac{1}{2}x \right)} = \cot \left(\tfrac{1}{2}x \right)$$

$$1 + (y')^2 = 1 + \cot^2 \left(\tfrac{1}{2}x \right) = \csc^2 \left(\tfrac{1}{2}x \right)$$

$$s = \int_{\frac{\pi}{3}}^{\pi} \sqrt{\csc^2 \left(\tfrac{1}{2}x \right)} \, dx = \int_{\frac{\pi}{3}}^{\pi} \left| \csc \left(\tfrac{1}{2}x \right) \right| dx$$

$$= \int_{\frac{\pi}{3}}^{\pi} \csc \left(\tfrac{1}{2}x \right) dx = 2 \ln \left| \csc \left(\tfrac{1}{2}x \right) - \cot \left(\tfrac{1}{2}x \right) \right| \Big|_{\frac{\pi}{3}}^{\pi}$$

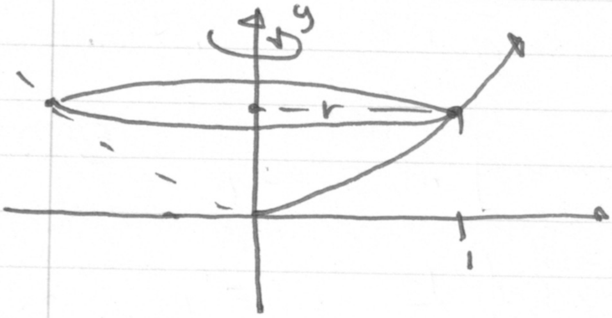
$$s = 2 \ln \left| \overset{1}{\cancel{\csc \left(\tfrac{1}{2}\pi \right)}} - \overset{0}{\cancel{\cot \left(\tfrac{1}{2}\pi \right)}} \right|$$

$$- 2 \ln \left| \overset{2}{\cancel{\csc \left(\tfrac{\pi}{6} \right)}} - \overset{\sqrt{3}}{\cancel{\cot \left(\tfrac{\pi}{6} \right)}} \right| \qquad s = -2 \ln (2 - \sqrt{3}$$

$$s = 2 \ln \overset{0}{\cancel{|1 - 0|}} - 2 \ln |2 - \sqrt{3}|$$

④ Surface Area

$y = x^2$; $0 \le x \le 1$; y-axis



$SA = \int 2\pi r \, ds$

x

$ds = \sqrt{1+(y')^2} \, dx$

$y' = 2x$; $1+(y')^2 = 1+4x^2$

$ds = \sqrt{1+4x^2}$; $SA = \int_0^1 2\pi x \sqrt{1+4x^2} \, dx$

$SA = 2\pi \int_0^1 x\sqrt{1+4x^2} \, dx$; $u = 1+4x^2$

$\frac{du}{dx} = 8x$;

$= 2\pi \int \cancel{x} \sqrt{u} \, \frac{du}{8\cancel{x}}$

$dx = \frac{du}{8x}$

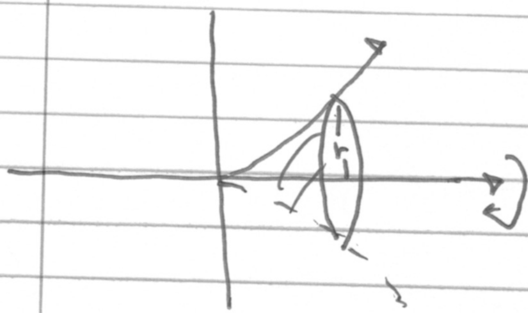
$\frac{\pi}{4} \int u^{1/2} \, du = \frac{\pi}{4} \frac{u^{3/2}}{3/2} = \frac{\pi}{\cancel{4}} \cdot \frac{2}{3} u^{3/2}$)

$= \frac{2\pi}{12} u^{3/2} = \frac{\pi}{6} (1+4x^2)^{3/2} \Big|_{x=0}^{x=1}$

$\frac{\pi}{6}(1+4)^{3/2} - \frac{\pi}{6}(1+0)$

$\frac{\pi}{6} \cdot 5^{3/2} - \frac{\pi}{6} = \boxed{\frac{\pi}{6}\left[5\sqrt{5} - 1\right]}$

(5) $y = x^2$; $0 \le x \le 1$; x- axis



$$SA = \int 2\pi \underset{y}{r}\, ds$$

$$\sqrt{1+(y')^2} = \sqrt{1+4x^2}$$

$$SA = \int_0^1 2\pi x^2 \sqrt{1+4x^2}\, dx \quad ; \quad n = 6$$

let $f(x) = x^2\sqrt{1+4x^2}$

so $S_6 = 2\pi \dfrac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2)$

$\qquad + 4f(x_3) + 2f(x_4)$

$\qquad + 4f(x_5) + f(x_6) \Big]$

note $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{6} = \dfrac{1}{6}$

0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$
x_0	x_1	x_2	x_3	x_4	x_5	x_6

$$S_6 = \frac{2\pi}{6\cdot 3}\Big[f(0) + 4f(\tfrac{1}{6}) + 2f(\tfrac{1}{3})$$
$$+ 4f(\tfrac{1}{2}) + 2f(\tfrac{2}{3}) + 4f(\tfrac{5}{6})$$
$$+ f(1)\Big]$$

$$S_6 = \frac{\pi}{9}\Big[f(0) + 4f(\tfrac{1}{6}) + 2f(\tfrac{1}{3}) + 4f(\tfrac{1}{2})$$
$$+ 2f(\tfrac{2}{3}) + 4f(\tfrac{5}{6}) + f(1)\Big]$$

$$\boxed{S_6 \approx 3.810}$$

(6)

$$L = \int_0^2 \sqrt{(x')^2 + (y')^2}\, dt$$

$$x = \frac{t}{1+t} \quad ; \quad x' = \frac{(1+t)\cdot 1 - 1\cdot t}{(1+t)^2}$$

$$= \frac{1+t-t}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$y = \ln(1+t) \quad ; \quad y' = \frac{1}{1+t}$$

$$(x')^2 + (y')^2 = \left[\frac{1}{(1+t)^2}\right]^2 + \left(\frac{1}{(1+t)}\right)^2$$

$$= \frac{1}{(1+t)^4} + \frac{1}{(1+t)^2}$$

$$= \frac{1}{(1+t)^4} + \frac{(1+t)^2}{(1+t)^4}$$

ie, $\sqrt{\frac{(1+t)^2+1}{(1+t)^4}} = \frac{\sqrt{(1+t)^2+1}}{\sqrt{(1+t)^4}}$

$$= \frac{\sqrt{(1+t)^2+1}}{(1+t)^2} \quad ; \quad \frac{\sqrt{(t+1)^2+1}}{(t+1)^2}$$

let $\cancel{x}$ $\int_0^2 \frac{\sqrt{(t+1)^2+1}}{(t+1)^2}\, dt$

let $u = t+1$; $du = dt$

$$= \int_1^3 \frac{\sqrt{u^2+1}}{u^2}\, du$$ (See past exam, notes, Hw)

$$= -\frac{\sqrt{u^2+1}}{u} + \ln\left(u + \sqrt{u^2+1}\right)\Bigg|_{u=1}^{u=3}$$

$$= -\cancel{1/1}\frac{\sqrt{9+1}}{3} + \ln\left(3+\sqrt{9+1}\right)$$

$$+ \cancel{1/1}\sqrt{2} - \ln(1+\sqrt{2})$$

$$- \frac{\sqrt{10}}{3} + \ln(3+\sqrt{10}) + \sqrt{2} - \ln(1+\sqrt{2})$$

$$\boxed{\ln\left(\frac{3+\sqrt{10}}{1+\sqrt{2}}\right) + \sqrt{2} - \frac{\sqrt{10}}{3}}$$

(7) $x = 10 - t^2$
$y = t^3 - 12t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

(HT) $\frac{dy}{dt} = 0$; $3t^2 - 12 = 0$
$3t^2 = 12$
$t^2 = 4$; $t = \pm 2$

($t = 2$) $x = 10 - 2^2$
$= 10 - 4$
$= 6$

$y = 2^3 - 12 \cdot 2$
$= 8 - 24$
$= -16$

$$\boxed{(6, -16)}$$

($t = -2$) $x = 10 - (-2)^2$
$= 10 - 4$
$= 6$

$y = (-2)^3 - 12(-2)$
$= -8 + 24$
$= 16$

$$\boxed{(6, 16)}$$

(VT) $\frac{dx}{dt} = 0$; $-2t = 0$; $t = 0$

$x = 10 - 0^2$
$x = 10$

$y = 0^3 - 12 \cdot 0$
$y = 0$

$$\boxed{(10, 0)}$$