Partial Fraction Decomposition Examples

ex $\int \frac{x-1}{x^2+3x+2}\,dx$ case (1)

factor

$(x+2)(x+1)$

$= \int \frac{x-1}{(x+2)(x+1)}\,dx$

pfd

$\frac{x-1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$

Distinct linear Factors

↑ add

$= \frac{A(x+1) + B(x+2)}{(x+2)(x+1)}$

ie, $\frac{x-1}{(x+2)(x+1)} = \frac{A(x+1) + B(x+2)}{(x+2)(x+1)}$

$\rightarrow \boxed{x-1 = A(x+1) + B(x+2)}$

let $x = -1$

$-1-1 = A(-1+1) + B(-1+2)$

$A(-1+1) = 0$

$-2 = B$

let $x = -2$

$$-2-1 = A(-2+1) + B(-2+2)$$

$$-3 = -A \quad \text{or} \quad A = 3$$

$$\int \frac{x-1}{(x+2)(x+1)}\,dx$$

$$= \int \left(\frac{3}{x+2} + \frac{-2}{x+1} \right) dx$$

$$= 3\int \frac{1}{x+2}\,dx - 2\int \frac{1}{x+1}\,dx$$

$\ln|x+2|$ $\ln|x+1|$

$$3\ln|x+2| - 2\ln|x+1| + C$$

But, by properties of logs

$$\ln|x+2|^3 - \ln|x+1|^2 + C$$

$$\ln\left[\left|\frac{(x+2)^3}{(x+1)^2}\right|\right] + C$$

ex $\int \frac{x^3 - 4x - 10}{x^2 - x - 6}\, dx$

factor

$(x-3)(x+2)$

$$\frac{x^3 - 4x - 10}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

Distinct linear factors

add

$$\frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$

So,

$$\frac{x^3 - 4x - 10}{(x-3)(x+2)} = \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$

$$x^3 - 4x - 10 = A(x+2) + B(x-3)$$

let $x = -2$

$$(-2)^3 - 4(-2) - 10 = \cancel{A(-2+2)}^{0} + B(-2-3)$$

$$-8 + 8 - 10 = -5B \quad ; \quad -10 = -5B$$

$$B = 2$$

let $x = 3$

$$3^3 - 4 \cdot 3 - 10 = A(3+2) + \cancel{B(3-3)}^{0}$$

$27 - 12 - 10 = 5A \quad ; \quad 5 = 5B$

$A = 1$

So,

$$\int \frac{x^3 - 4x - 10}{(x-3)(x+2)}\,dx = \int \left(\frac{1}{x-3} + \frac{2}{x+2}\right) dx$$

$$= \int \frac{1}{x-3}\,dx + \int \frac{2}{x+2}\,dx$$

$$= \int \frac{1}{x-3}\,dx + 2\int \frac{1}{x+2}\,dx$$

$$\boxed{\ln|x-3| + 2\ln|x+2| + C}$$

or $\ln|x-3| + \ln|x+2|^2 + C$

$\ln|x-3| + \ln(x+2)^2 + C$

$$\boxed{\ln\left[|x-3|(x+2)^2\right] + C}$$

class example (Repeat linear Factors)

ex $\int \frac{1}{x^2(x-1)^2}\,dx$

$= \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}\,dx$

$1 \equiv Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$

$x=0$; $1 = B(-1)^2$; $\boxed{B=1}$

$x=1$; $1 = D \cdot 1^2$; $\boxed{D=1}$

what about C and D ?

$x=2$ or any other value.

$1 = A\cdot 2(2-1)^2 + B(2-1)^2 + C\cdot 2^2(2-1) + D\cdot 2^2$

$B \to 1$; $D \to 1$

$1 = 2A + 1 + 4C + 4$

$1 = 2A + 4C + 5$; $\boxed{2A + 4C = -4}$

or $\boxed{A + 2C = -2}$

let $x=-2$

$1 = A(-2)(-2-1)^2 + B(-2-1)^2 + C(-2)^2(-2-1)$

$+ D(-2)^2$

$1 = -2A \cdot 9 + 9B + 4C \cdot (-3) + 4D$

9 9

1 1

$1 = -18A + 9 - 12C + 4$

$1 = -18A - 12C + 13$; $\boxed{-12 = -18A - 12C}$

or

$-2 = -3A - 2C$

or $\boxed{3A + 2C = 2}$

we have 2 eq + 2 unknows!

$$\begin{cases} 3A + 2C = 2 \\ A + 2C = -2 \end{cases}$$ add method

$3A + 2C = 2$

$-A - 2C = 2$

$+$

$2A = 4$

$2A = 4$; $\boxed{A = 2}$

use back sub

$A + 2C = -2$

2

$2 + 2C = -2$

$2C = -4$; $\boxed{C = -2}$

$$\int \left(\frac{2}{x} + \frac{1}{x^2} - \frac{2}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$2\int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx - 2\int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$2\ln|x| - \frac{1}{x} - 2\ln|x-1| - \frac{1}{x-1} + C$$

ex: $\int \frac{2x+4}{x^3-2x^2}\,dx = \int \frac{2x+4}{x^2(x-2)}\,dx$

$x^2(x-2)$

LCM

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

repeat linear $x \cdot x$

add

$$= \frac{Ax(x-2)}{LCM} + \frac{B(x-2)}{LCM} + \frac{Cx^2}{LCM}$$

i.e., $2x+4 = Ax(x-2) + B(x-2) + Cx^2$;

let $x = 0$;

$2 \cdot 0 + 4 = B(0-2)$; $4 = -2B$; $B = -2$

let $x = 2$

$2 \cdot 2 + 4 = C \cdot 2^2 \quad ; \quad 8 = 4C \; ; \quad C = 2$

let $x = 1$

$2 \cdot 1 + 4 = A \cdot 1 (1-2) + B(1-2) + C \cdot 2$

$6 = -A - B + C$

$-2 \qquad 2$

$6 = -A - (-2) + 2 \; ; \quad 6 = -A + 2 + 2$

$6 = -A + 4 \; ; \quad -A = 2 \; ; \quad A = -2$

$$\int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \right) dx$$

$$\int \left(\frac{2}{x} - \frac{2}{x^2} + \frac{2}{x-2} \right) dx$$

$$= -2 \int \frac{1}{x} dx - 2 \int \frac{1}{x^2} dx + 2 \int \frac{1}{x-2} dx$$

$$-2 \ln|x| + 2\frac{1}{x} + 2 \ln|x-2| + C$$

$$2 \left[\ln|x-2| - \ln|x| \right] + \frac{2}{x} + C$$

$$\boxed{2 \ln \left| \frac{x-2}{x} \right| + \frac{2}{x} + C}$$