Improper Integral Examplos

Infite Intervals

ex $\int_0^{\infty} \frac{x}{(x^2+2)^2}\,dx$ case (1)

$\lim_{t\to\infty} \int_0^t \frac{x}{(x^2+2)^2}\,dx$;

$I(t) = \int_0^t \frac{x}{(x^2+2)^2}\,dx$

u-sub

$u = x^2+2$

$\frac{du}{dx} = 2x$;

$dx = \frac{du}{2x}$

$= \int \frac{\cancel{x}}{u^2} \frac{du}{2\cancel{x}}$

$= \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2u}\Big|_0^t$

$= -\frac{1}{2(x^2+2)}\Big|_0^t = -\frac{1}{2(t^2+2)} + \frac{1}{2(0^2+2)}$

$I(t) = \frac{1}{4} - \frac{1}{2(t^2+2)}$

as $t\to\infty$

$I(t) \to \boxed{\frac{1}{4}}$

converjes

ex $\int_0^{\infty} \frac{1}{x^2+3x+2}\,dx$ case (1)

$\lim_{t\to\infty} \int_0^t \frac{1}{x^2+3x+2}\,dx$

$$I(t) = \int_0^t \frac{1}{(x+2)(x+1)}\,dx$$

note $x \neq -2$; $x \neq -1$

but, $-2 \notin [0, t]$; $-1 \notin [0, t]$

use partial fraction Decomposition!
(Pfd)

$$\frac{1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$\rightarrow \quad 1 = A(x+1) + B(x+2) \quad ; \quad A = -1$$
$$B = 1$$

$$I(t) = \int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx$$

$$= \ln|x+1| - \ln|x+2|$$

$$= \ln \left| \frac{x+1}{x+2} \right|_0^t$$

$$= \ln\left(\frac{t+1}{t+2}\right) - \ln\left|\frac{0+1}{0+2}\right|$$

$$I(t) = \ln\left(\frac{t+1}{t+2}\right) - \ln(1/2)$$

as $t \to \infty$;

$$\frac{t+1}{t+2} = \frac{1 + 1/t}{1 + 2/t} \rightarrow 1$$

so, $\ln\left|\dfrac{1+1/t}{1+2/t}\right| \to \ln|1| = 0$

ie as $t\to\infty$; $I(t) \to 0 - \ln(1/2)$

$$\boxed{-\ln(1/2)}$$

converges

ex $\int_0^{\infty} \cos^2\theta \, d\theta$ case ①

$$\lim_{t\to\infty}\int_0^t \cos^2\theta \, d\theta$$

$$\frac{1}{2}[1 + \cos(2\theta)]$$

$$I(t) = \int_0^t \frac{1}{2}[1+\cos(2\theta)]d\theta$$

$$= \frac{1}{2}\int_0^t d\theta + \frac{1}{2}\int \cos(2\theta)\, d\theta$$

u-sub $u = 2\theta$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\Big|_0^t$$

$$= \frac{1}{2}t + \frac{1}{4}\sin(2t) - \left(\cancel{\frac{1}{2}\cdot 0}^{\,0} + \cancel{\frac{1}{4}\sin(2\cdot 0)}^{\,0}\right)$$

$I(t) = \frac{1}{2}t + \frac{1}{4}\sin(2t)$ as $t\to\infty$

$I(t) \to \infty$

why ?

$|\sin(2t)| \le 1$ all t

for $t > 0$; $\sin(2t) \le 1$

Since $I(t) = \frac{1}{2}t + \frac{1}{4}\sin(2t)$

we have $I(t) \le \frac{1}{2}t + \frac{1}{4}$

as $t \to \infty$; $\frac{1}{2}t \to \infty$

or $I(t) \to \boxed{\infty}$

Diverges.

Case (1)

ex $\int_1^\infty \frac{1}{x}\,dx$ case (1) This is an important improper integral

$\lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$;

$I(t) = \int_1^t \frac{1}{x}\,dx$

$= \ln|x| \Big|_1^t = \ln|t| - \cancel{\ln|1|}^{\,0}$

$I(t) = \ln|x|$ as $t \to \infty$; $\ln(t) \to \boxed{\infty}$

Divergence

$\underline{ex}$ $\int_{-\infty}^{0} x e^x \, dx$ case (2)

$$\lim_{t \to -\infty} \int_t^0 x e^x \, dx$$

$I(t) = \int_t^0 x e^x \, dx$ IBP

$$= x e^x - e^x \Big|_t^0$$

$$= 0e^0 - e^0 - (te^t - e^t)$$

$I(t) = -1 - te^t - e^t$ ∴ as $t \to -\infty$

$$\lim_{t \to -\infty} (-1 - te^t - e^t)$$

$$-1 - \lim_{t \to -\infty} te^t - \cancel{\lim_{t \to -\infty}} e^t \quad 0$$

$-\infty \cdot 0$

IP

$\lim_{t \to -\infty} \frac{t}{e^{-t}}$ $\frac{-\infty}{\infty}$ IQ

$$\overset{=}{H} \lim_{t \to -\infty} \frac{1}{-e^{-t}} = \lim_{t \to -\infty} -e^t$$

$$-\lim_{t \to -\infty} e^{t}$$

$$-1 \cdot 0$$

$$0$$

ie, as $t \to \infty$; $I(t) \to \boxed{-1}$

convergence

ex $\displaystyle\int_{-\infty}^{-1} e^{-2x}\,dx$

$$\lim_{t \to -\infty} \int_{t}^{-1} e^{-2x}\,dx$$

$$I(t) = \int_{t}^{-1} e^{-2x}\,dx$$

u-sub

$u = -2x$

$du = -2\,dx$; $dx = \dfrac{du}{-2}$

$$= \int e^{u} \frac{du}{-2} = -\frac{1}{2}\int e^{u} = -\frac{1}{2} e^{u}$$

$$= -\frac{1}{2} e^{-2x} \Big|_{t}^{-1}$$

$$= -\frac{1}{2} e^{-2(-1)} + \frac{1}{2} e^{-2t}$$

$I(t) = -\frac{1}{2} e^{2} + \frac{1}{2} e^{-2t}$ as $t \to \infty$

note $e^{-2t} \to \infty$ $I(t) \to \infty$; Diverges ex $\int_{-\infty}^{\infty} e^{-|x|} \, dx$

$$= \int_{-\infty}^{0} e^{-|x|} \, dx + \int_{0}^{\infty} e^{-|x|} \, dx$$

$$|x| = \begin{cases} x & ; \ x \geq 0 \\ -x & ; \ x < 0 \end{cases}$$

$$= \int_{-\infty}^{0} e^{-(-x)} \, dx \ \ (x < 0) + \int_{0}^{\infty} e^{-x} \, dx \ \ (x \geq 0)$$

$$= \int_{-\infty}^{0} e^{x} \, dx + \int_{0}^{\infty} e^{-x} \, dx$$

case (2) case (1)

$$\lim_{t \to -\infty} \int_{t}^{0} e^{x} \, dx \ ;$$

$$I(t) = \int_{t}^{0} e^{x} \, dx$$

as $t \to -\infty$

$I(t) \to \boxed{1}$

$$= e^{x} \Big|_{t}^{0}$$

$$= e^{0} - e^{t}$$

$$= 1 - e^{t}$$

Now,

$$\int_0^{\infty} e^{-x}\,dx$$

$$= \lim_{t\to\infty} \int_0^{t} e^{-x}\,dx$$

$$I(t) = \int_0^{t} e^{-x}\,dx = -e^{-x}\Big|_0^{t}$$

$$= -e^{-t} + \cancel{e^{-1\cdot 0}}\ 1$$

$$I(t) = 1 - e^{-t}$$ as $t \to \infty$

$I(t) \to 1$

ie, $$\int_{-\infty}^{\infty} e^{-|x|}\,dx = 1 + 1$$

$$= \boxed{2}$$ Convergence

Type II Infinite Discontinuities

ex: $$\int_0^{3} \frac{1}{x\sqrt{x}}\,dx$$; $f(x)$ is discontinuous at $x=0$; Left end point $[0,3]$ case 2

$$= \lim_{t\to 0^{+}} \int_t^{3} \frac{1}{x\sqrt{x}}\,dx$$

$$I(t) = \int_t^3 \frac{1}{x \cdot x^{1/2}}\,dx = \int_t^3 x^{-3/2}\,dx$$

$$= \frac{x^{-3/2+1}}{-\frac{3}{2}+1} = \frac{x^{-1/2}}{-1/2} = -\frac{2}{\sqrt{x}}$$

$$I(t) = -\frac{2}{\sqrt{x}}\Big|_t^3$$

$$I(t) = -\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{t}} \; ; \; I(t) = \frac{2}{\sqrt{t}} - \frac{2}{\sqrt{3}}$$

as $t \to 0^+$; $I(t) \to \boxed{\infty}$

Diverges

<u>ex</u> $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$; $f(x)$ is discontinuous at $x = 1$; $[0,1]$

Right endpoint case 1

$$= \lim_{t \to 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}}\,dx$$

$$I(t) = \int_0^t \frac{1}{\sqrt{1-x^2}}\,dx$$

as $t \to 1^-$

$I(t) \to \boxed{\frac{\pi}{2}}$ converges

$$= \sin^{-1}(x)\Big|_0^t$$

$$= \sin^{-1}(t) - \cancel{\sin^{-1}(0)}^{\,0}$$

$$\lim_{t \to 1^-} \sin^{-1}(t)$$

$$I(t) = \sin^{-1}(t)$$

$$= \sin^{-1}(1) = \frac{\pi}{2}$$

$$= \frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3|$$

$$= \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| \Bigg|_0^t$$

$$= \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| - \frac{1}{5}\ln\left|\frac{0-2}{0+3}\right|$$

$$= \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| - \frac{1}{5}\ln\left|-\frac{2}{3}\right|$$

$$I(t) = \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| - \frac{1}{5}\ln(2/3)$$

as $t \to 2^-$; $I(t) \to \boxed{-\infty}$

$$\lim_{t\to 2^-}\left[\frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| - \frac{1}{5}\ln(2/3)\right]$$

$$\frac{1}{5}\boxed{\lim_{t\to 2^-}\ln\left|\frac{t-2}{t+3}\right|} - \frac{1}{5}\ln(2/3)$$

where does this go?

as $t \to 2^-$; $\frac{t-2}{t+3} \to 0$

$$\to \ln\left|\frac{t-2}{t+3}\right| \to -\infty$$

ie, $-\infty - \frac{1}{5}\ln(2/3) = -\infty$, Diverges

so, we do not need to evaluate

$$\int_2^4 \frac{1}{(x+3)(x-2)}\,dx \;!$$

$\underline{ex}$ $\displaystyle\int_0^4 \frac{1}{x^2+x-6}\,dx$ $\quad (x+3)(x-2)$

f(x) is discontinous at $x=-3$; $x=2$

$[0, 4]$ case ③

$$= \int_0^2 \frac{1}{(x+3)(x-2)}\,dx + \int_2^4 \frac{1}{(x+3)(x-2)}\,dx$$

$[0, 2]$ — discontinuity case ①

$[2, 4]$ — discontinuity case ②

$$\lim_{t\to 2^-} \int_0^t \frac{1}{(x+3)(x-2)}\,dx$$

$$I_1 = \int_0^t \frac{1}{(x+3)(x-2)}\,dx \quad \text{use pdf}$$

$$\frac{1}{(x+3)(x-2)}\,dx = \frac{A}{x+3} + \frac{B}{x-2}$$

$$1 = A(x-2) + B(x+3) \quad ; \quad A = -\frac{1}{5} \quad B = \frac{1}{5}$$

$$\frac{1}{(x+3)(x-2)} = \frac{1/5}{x-2} - \frac{1/5}{x+3}$$

$$\int \frac{1}{(x+3)(x-2)}\,dx = \int\left(\frac{1/5}{x-2} - \frac{1/5}{x+3}\right)dx$$

$$= \frac{1}{5}\int \frac{1}{x-2}\,dx - \frac{1}{5}\int \frac{1}{x+3}\,dx$$

ex $\displaystyle\int_0^2 \frac{x-3}{2x-3}\,dx$ — f(x) is discontinuous at $x = 3/2$ which is in $[0, 2]$

case (3)

$$= \int_0^{3/2} \frac{x-3}{2x-3}\,dx + \int_{3/2}^{2} \frac{x-3}{2x-3}\,dx$$

case (1) — case (2)

$$\lim_{t \to \frac{3}{2}^-} \int_0^t \frac{x-3}{2x-3}\,dx$$

$$I(t) = \int \frac{x-3}{2x-3}\,dx$$

$$\begin{array}{r|l} & \ \tfrac{1}{2} \\ \hline 2x-3 & \ x-3 \\ & -x+3/2 \\ \hline & \quad -3/2 \end{array}$$

$$= \int_0^t \left(\frac{1}{2} - \frac{3/2}{2x-3}\right) dx$$

$$= \frac{1}{2}\int_0^t dx - \frac{3}{2}\int_0^t \frac{1}{2x-3}\,dx$$

$$= \frac{1}{2}x - \frac{3}{4}\ln|2x-3| \Big|_0^t$$

$$= \frac{1}{2}t - \frac{3}{4}\ln|2t-3| - \left[\cancelto{0}{\tfrac{1}{2}\cdot 0} - \frac{3}{4}\cancelto{\ln(3)}{\ln|2\cdot 0-3|}\right]$$

$$I(t) = \frac{1}{2}t - \frac{3}{4}\ln|2t-3| + \frac{3}{4}\ln(3)$$

as $t \to \frac{3}{2}^-$

$I(t) \to$? ∞ Diverges

$$\lim_{t \to \frac{3}{2}^-} \left[\frac{1}{2}t - \frac{3}{4} \ln|2t-3| + \frac{3}{4}\ln(3) \right]$$

$$3 - \frac{3}{4} \boxed{\lim_{t \to 3/2^-} \ln|2t-3|} + \frac{3}{4}\ln(3)$$

what is this ?

as $t \to \frac{3}{2}^-$; $|2t-3| \to 0$

$\therefore \ln|2t-3| \to -\infty$

ie, $3 - \frac{3}{4}(-\infty) + \frac{3}{4}\ln(3)$

$= \infty$

Divergent