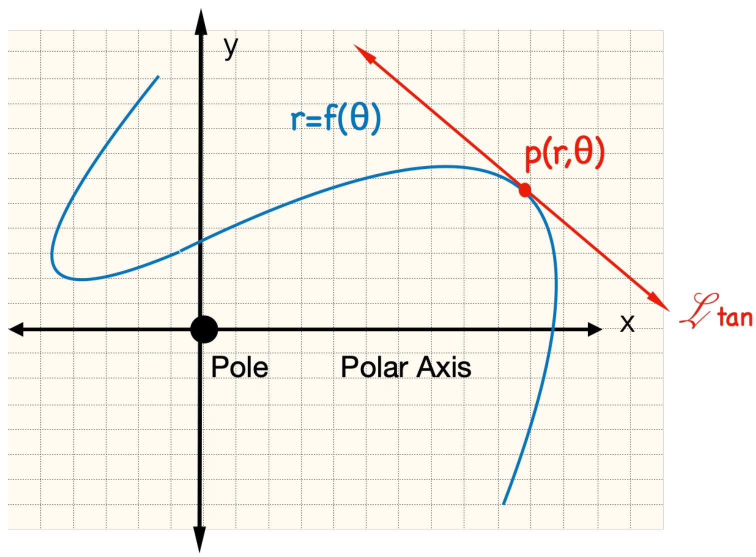
Calculus of Polar Equations

We want to find the Tangent Line $\ell_{\tan}$ to the Polar Curve $r = f(\theta)$. We just consider the following parametric equations and allow for the parameter to be θ as opposed to t.



$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r\sin(\theta))}{\frac{d}{d\theta}(r\cos(\theta))} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} \quad \text{if} \quad \frac{dx}{d\theta} \neq 0$$

Horizontal Tangents

Points on the curve in which $\frac{dy}{d\theta} = 0$, but $\frac{dx}{d\theta} \neq 0$

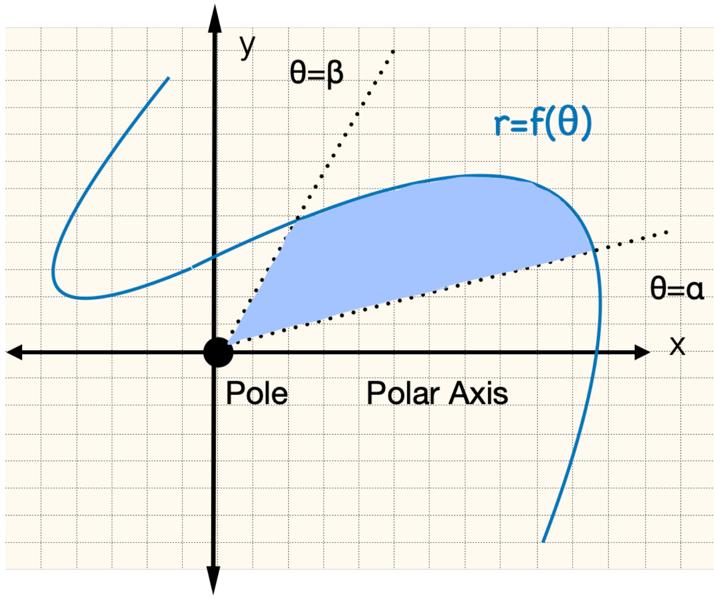
Vertical Tangents

Points on the curve in which $\frac{dx}{d\theta} = 0$, but $\frac{dy}{d\theta} \neq 0$

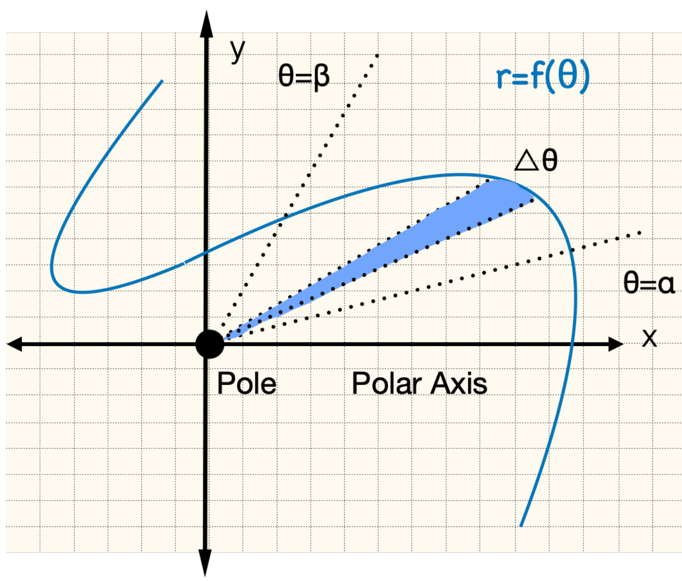
Tangent Line at the Pole

We let $r = 0$ and $\frac{dy}{dx} = \tan(\theta)$ if $\frac{dr}{d\theta} \neq 0$

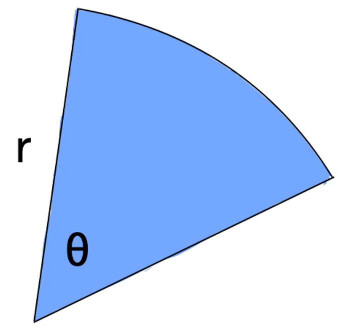
We would like to determine the area under the curve $r = f(\theta)$



We consider a partition of the interval $\alpha \leq \theta \leq \beta$ into n-subintervals of equal length $\Delta\theta$



Use the area of a sector formula $A = \frac{1}{2}r^2\theta$ to add the n sector areas.



where $A_i = \frac{1}{2}r^2{}_i\Delta\theta$ for $i = 1\ to\ n$

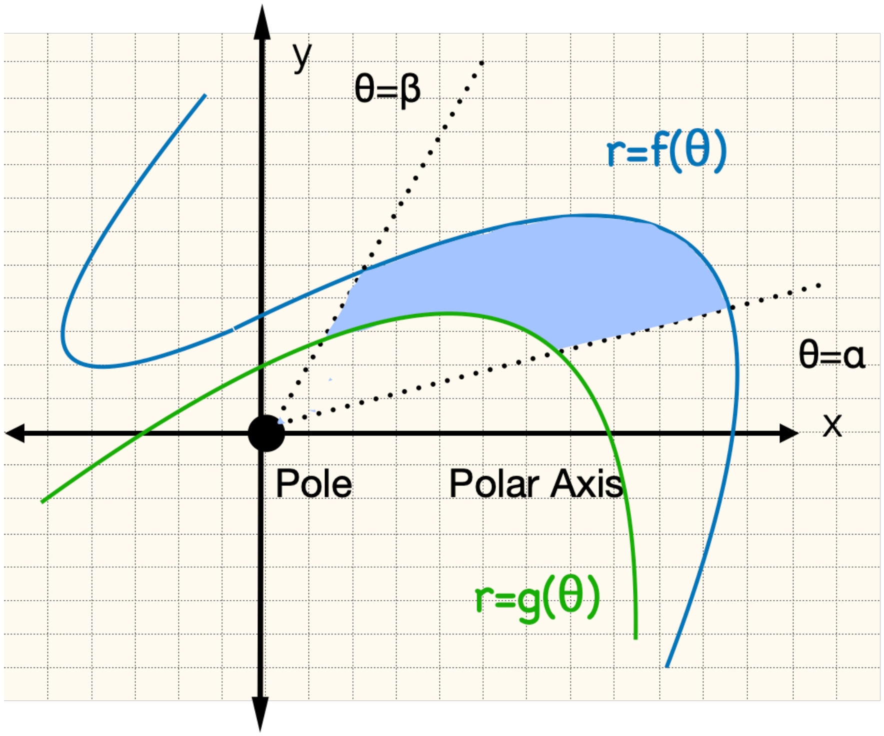
$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2\, \Delta\theta$$

as $n \to \infty$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2\Delta\theta$$

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 d\theta \qquad A = \int_{\alpha}^{\beta} \frac{1}{2}r^2 d\theta}$$

Area Between Two Curves



$$A = A_f - A_g$$

$$= \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2}[g(\theta)]^2 d\theta$$

$$= \int_{\alpha}^{\beta} \left\{\frac{1}{2}[f(\theta)]^2 - \frac{1}{2}[g(\theta)]^2\right\} d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} \{[f(\theta)]^2 - [g(\theta)]^2\} \, d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} ({r^2}_t - {r^2}_b) \, d\theta$$

Arc Length

Since $x = rcos(\theta) = f(\theta)\cos(\theta)$ and $y = rsin(\theta) = f(\theta)\sin(\theta)$ where $\alpha \leq \theta \leq \beta$ we get the following equations for their derivatives.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(rcos(\theta)) = \frac{dr}{d\theta}cos(\theta) - rsin(\theta)$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(rsin(\theta)) = \frac{dr}{d\theta}sin(\theta) + rcos(\theta)$$

These equations can be substituted into the arc length formula.

$$S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

We will need some algebra

$$\left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}cos(\theta) - rsin(\theta)\right)^2 = \left(\frac{dr}{d\theta}\right)^2 cos^2(\theta) - 2rsin(\theta)cos(\theta)\frac{dr}{d\theta} + r^2 sin^2(\theta)$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}sin(\theta) + rcos(\theta)\right)^2 = \left(\frac{dr}{d\theta}\right)^2 sin^2(\theta) + 2rsin(\theta)cos(\theta)\frac{dr}{d\theta} + r^2 cos^2(\theta)$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 cos^2(\theta) + r^2 sin^2(\theta) + \left(\frac{dr}{d\theta}\right)^2 sin^2(\theta) + r^2 cos^2(\theta)$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Now,

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$\boxed{S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 d\theta}}$$