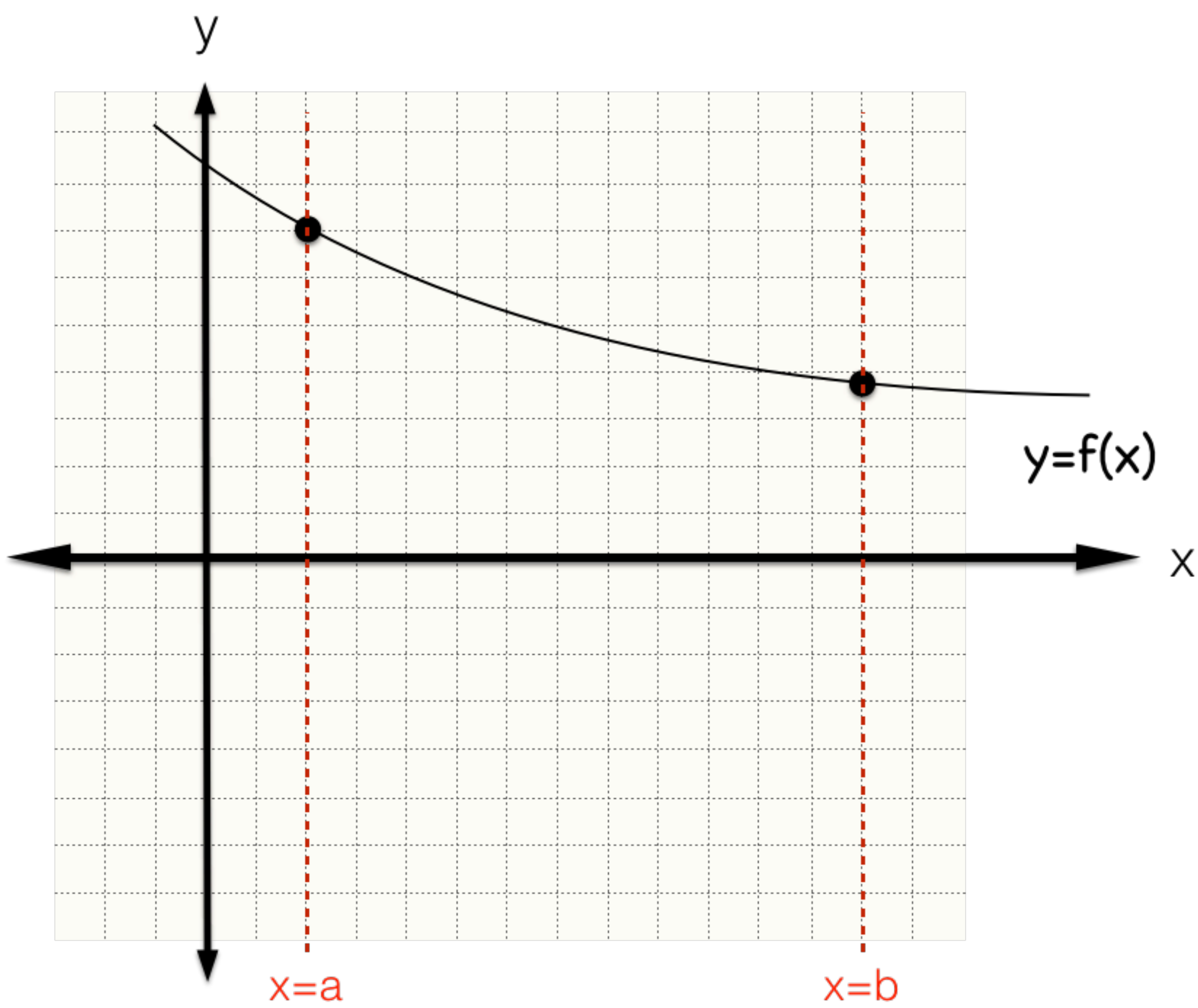
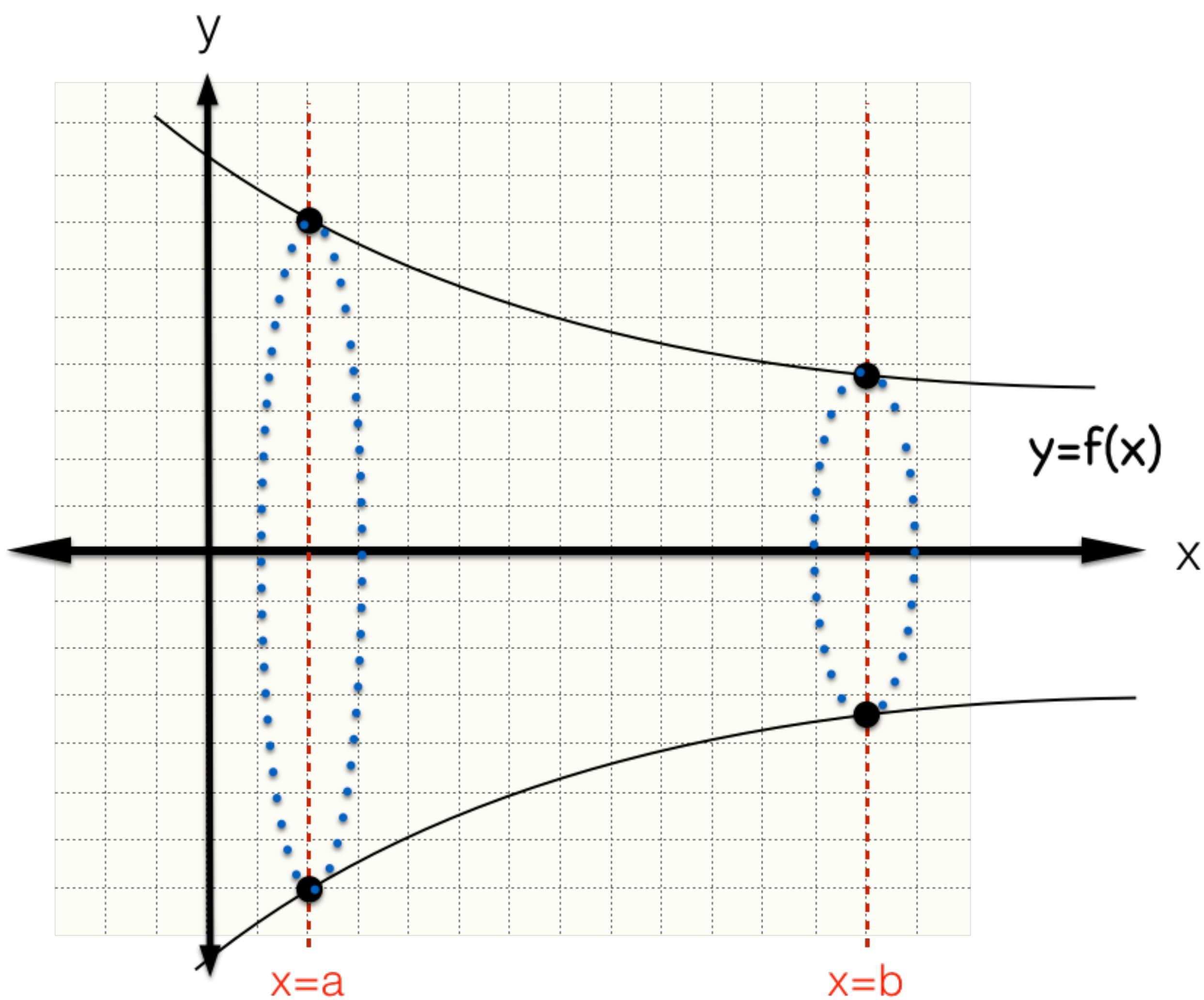
Surface Area for a Solid of Revolution

We start with a smooth continuous curve $y = f(x)$ over an interval $[a, b]$

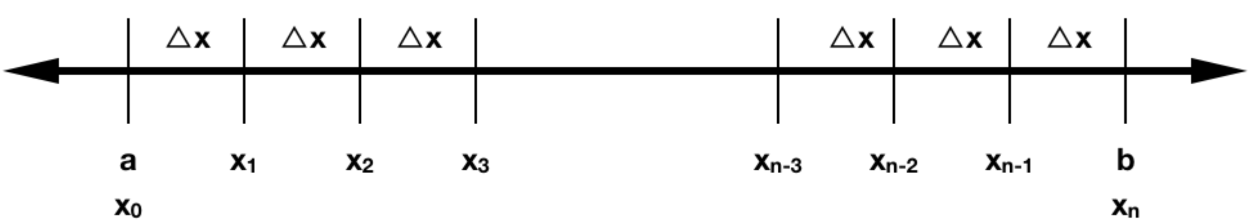
$$\boldsymbol{SA = \int 2\pi r ds}$$



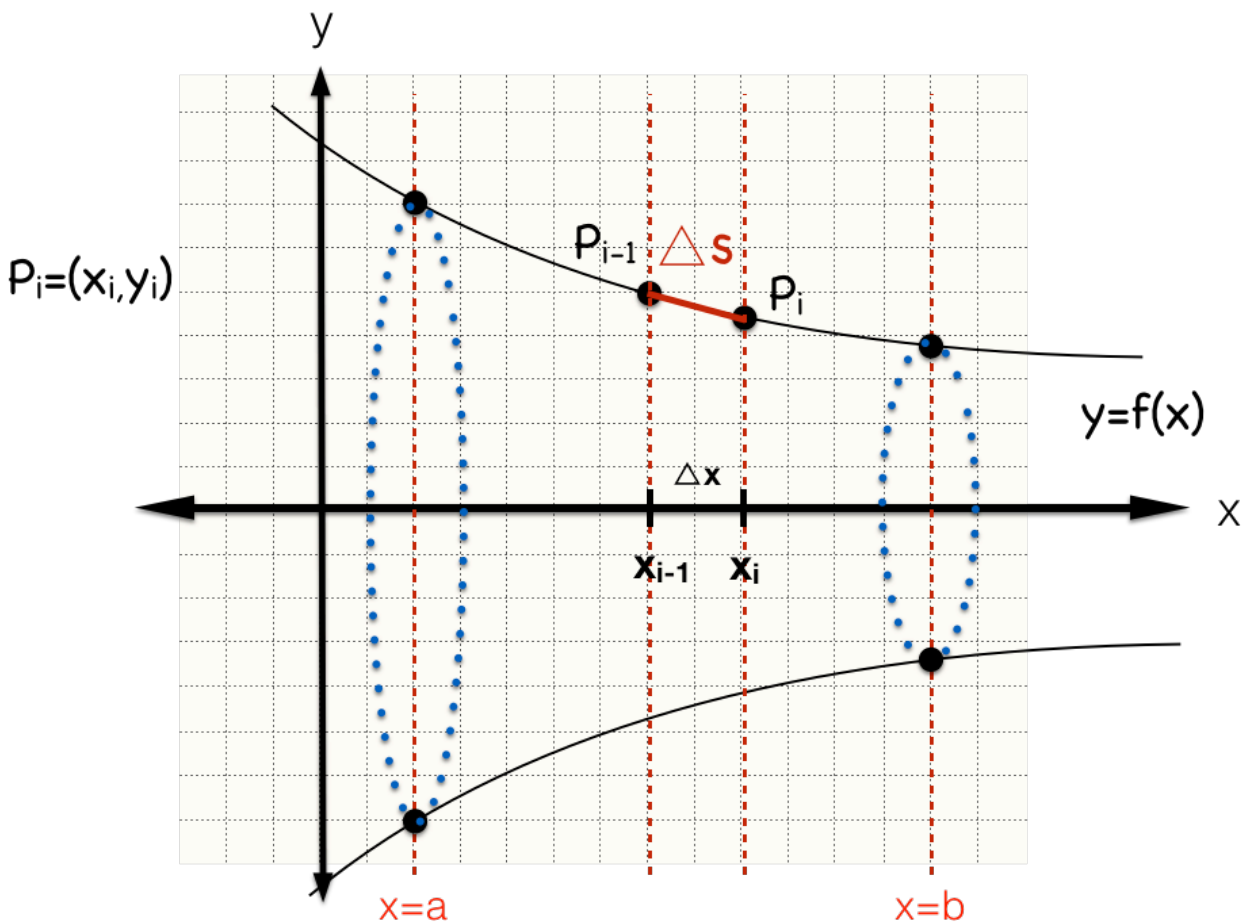
Rotate it about the x-axis.



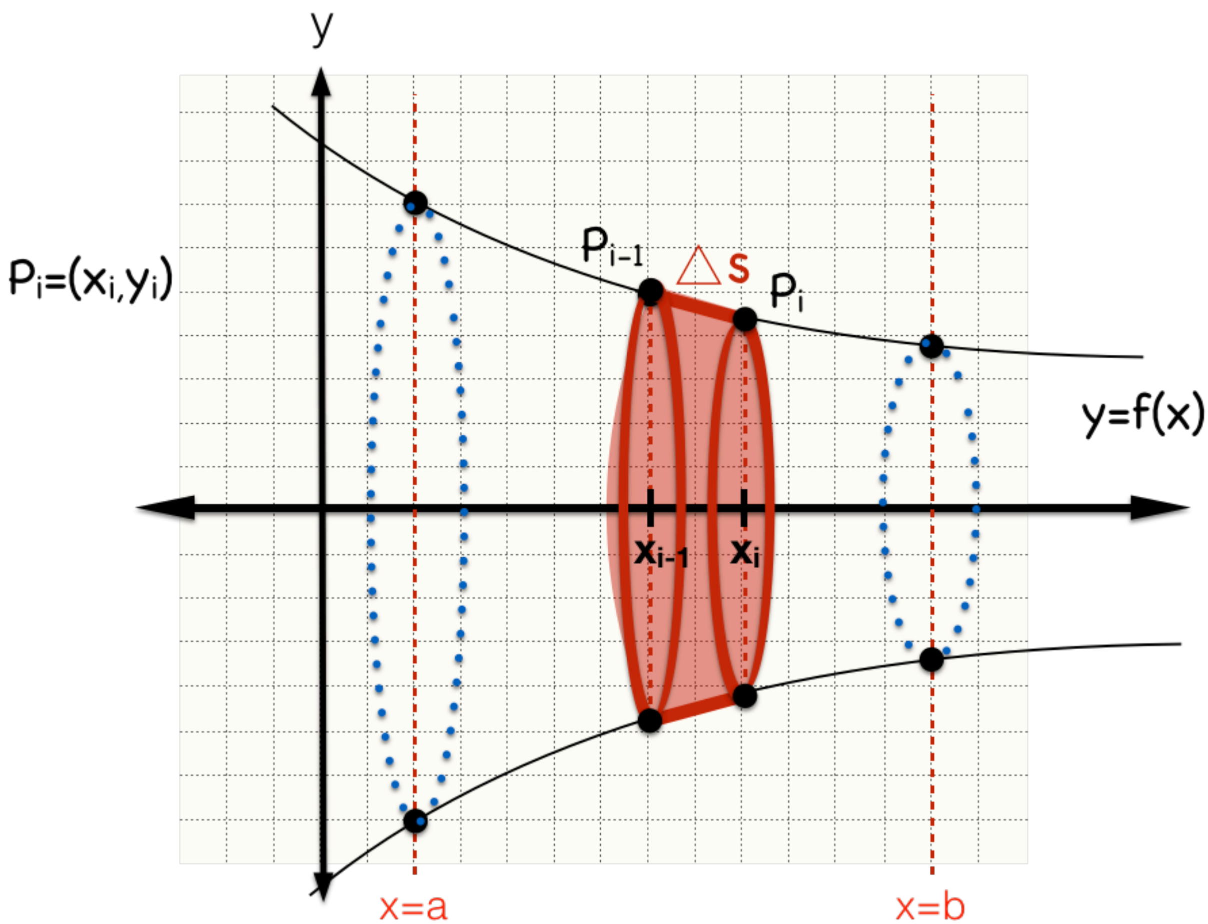
We partition the interval $[a, b]$ into n-subintervals of equal length Δx as before.



We now focus on the ith sub-interval $[x_{i-1}, x_i]$ and let $ds = |P_{i-1}P_i|$ represent the arc length along the curve from P_{i-1} to P_i.

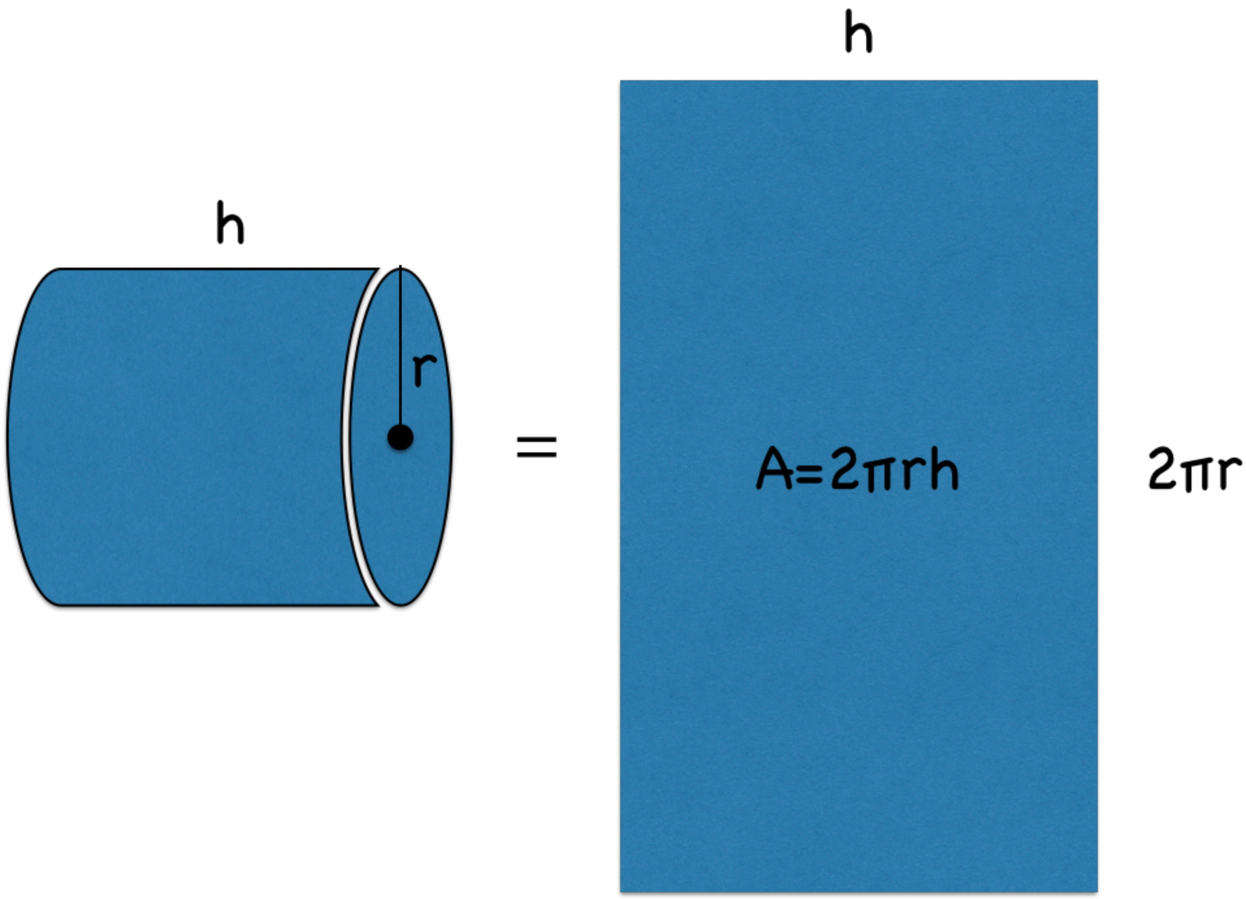


A more detail description of what we obtain by rotating the curve $y = f(x)$ about the x-axis is the ds line segment outlines a conical shape described below in red.

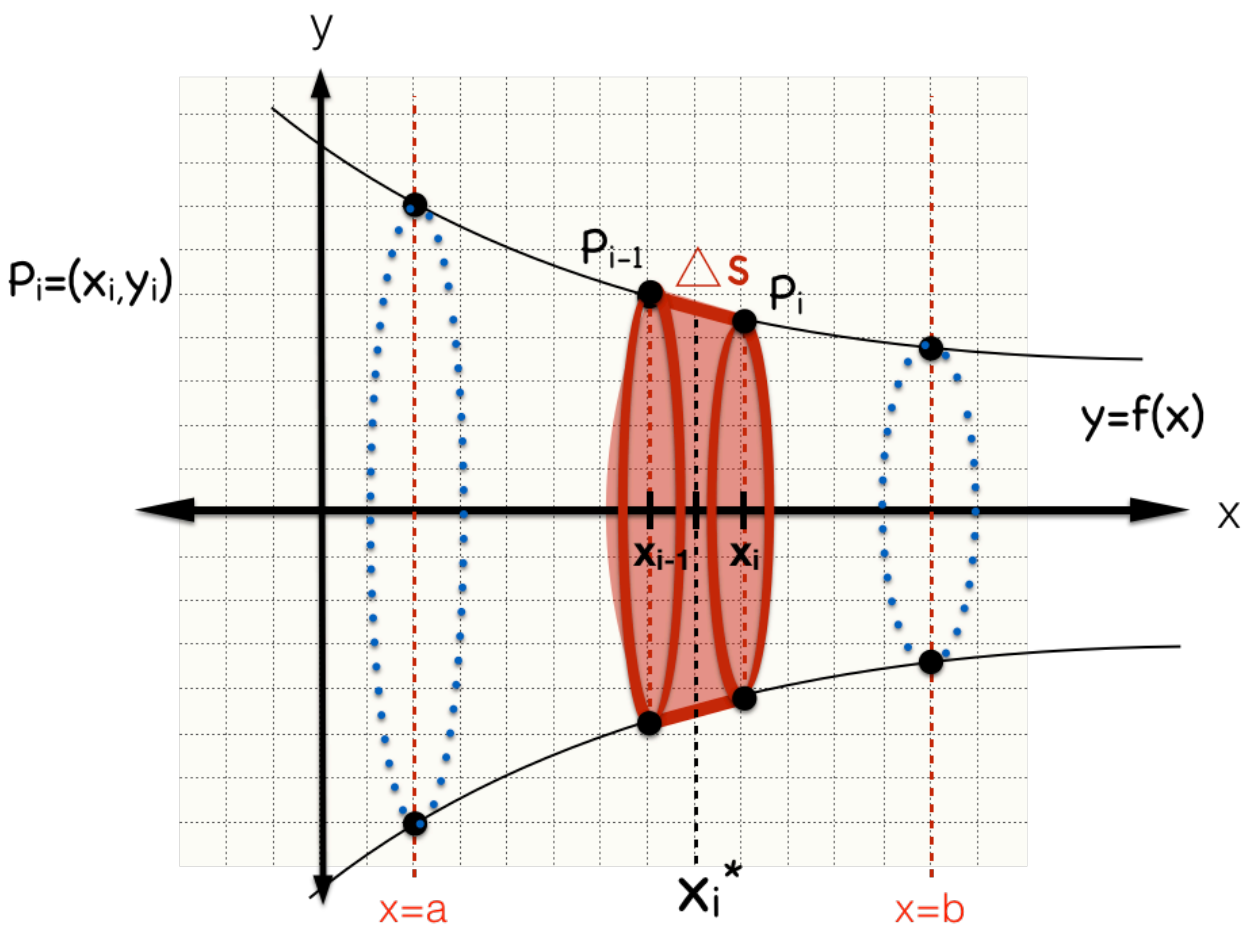


It resembles a coffee cup sieve that is used to keep your hands from getting burned while holding a hot cup of coffee.

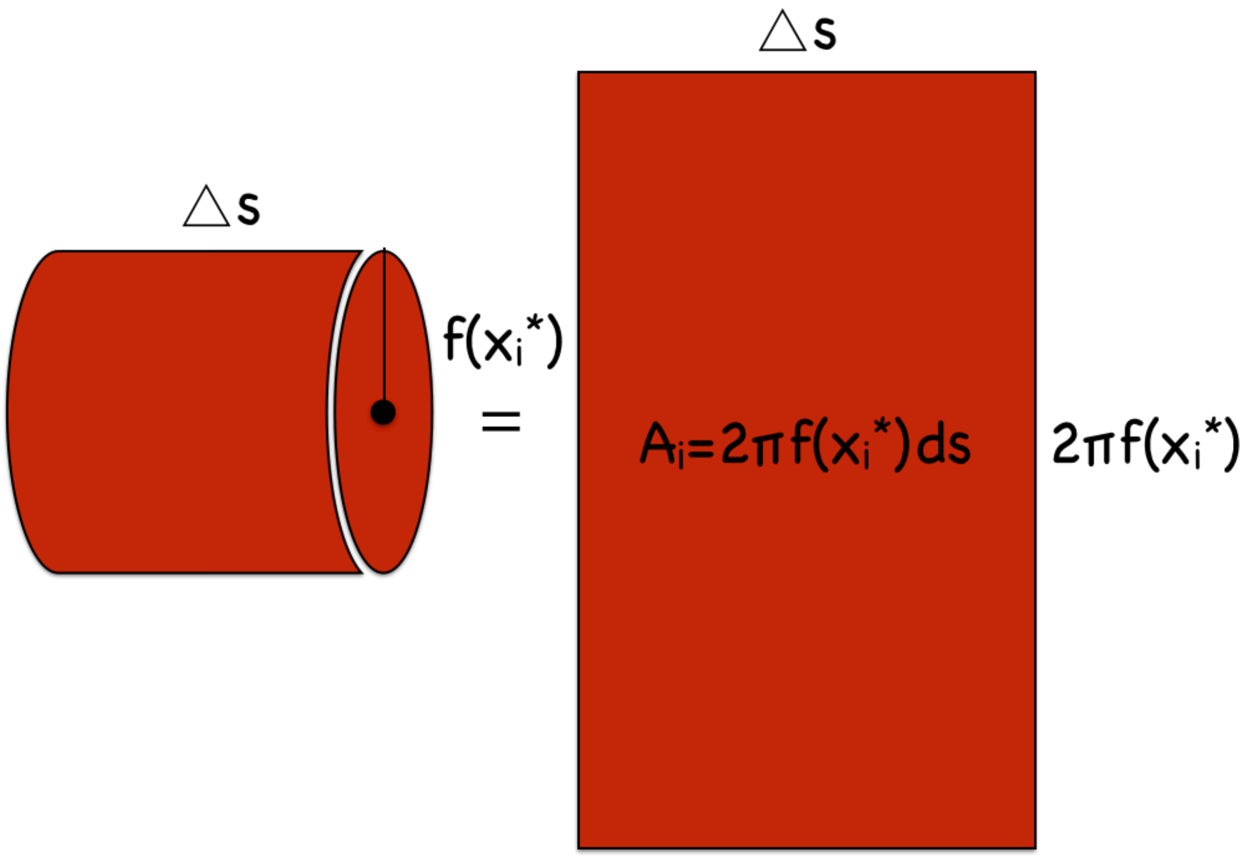
For very small Δx, and consequentially Δs, our conical shape resembles a cylindrical shape that can be cut along its side to create a rectangle, while ignoring the top and bottom of the cylinder. This is known as the lateral (side) surface area of the cylinder.



We will also be letting x_i^* be in $[x_{i-1}, x_i]$ for $i = 1\ to\ n$.



The Geometry can be used to fit our model in computing the lateral (side) surface area of a solid of revolution in letting $r = f(x_i^*)$ and the $h = ds$.



The lateral surface area can be approximated by doing the following summation.

$$SA \approx A_1 + A_2 + A_3 + \cdots + A_n$$

$$\approx \sum_{i=1}^{n} A_i$$

$$\approx \sum_{i=1}^{n} 2\pi f(x_i^*)\,\Delta s$$

Now, as $n \to \infty$ our approximation becomes exact as we obtain a Riemann Sum.

$$SA = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\,\Delta s$$

$$= \int_a^b 2\pi f\,(x) ds$$

An important potion of the formula is the term ds as your integrand $f(x)$ is a function of x and is a different variable.

$$\boldsymbol{SA = \int_a^b 2\pi y\, ds} \text{ where } ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx \;\; or \;\; ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$$

$$SA = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx \;\; ; \;\; SA = \int_a^b 2\pi y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx$$

Similarly, we can describe a curve as $x = g(y)$ over $c \leq y \leq d$ and the formula for the surface are becomes the following.

$$\boldsymbol{SA = \int 2\pi y ds} \; ; \; SA = \int_a^b 2\pi y \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$$

Rotate about the y-axis.

$$\boldsymbol{SA = \int 2\pi r x ds} \text{ where } ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx \;\; or \;\; ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$$