Probability and Multiple Selections

Starting with the definition of **conditional probability**, we can develop two very important probability formulas that are studied extensively and are practical. One is called the **Multiplication Rule for Probability** and the other is known as **Bayes Law**. I will engage in some Mathematics to develop (prove) both formulas.

$$P(A|B) = \frac{n(A\ and\ B)}{n(B)}$$

$$= \frac{n(A\ and\ B)}{n(B)} \cdot 1$$

$$= \frac{n(A\ and\ B) \cdot \frac{1}{n(S)}}{n(B) \cdot \frac{1}{n(S)}}$$

$$= \frac{\frac{n(A\ and\ B)}{n(S)}}{\frac{n(B)}{n(S)}}$$

$$= \frac{P(A\ and\ B)}{P(B)}$$

Using Algebra, we get the following equivalent statements.

$$P(A|B) = \frac{P(A\ and\ B)}{P(B)} \quad \leftrightarrow \quad P(A\ and\ B) = P(B)P(A|B)$$

I want to point out that the statement $\boldsymbol{A\ and\ B}$ is logically equivalent to the statement $\boldsymbol{B\ and\ A}$. The word $\boldsymbol{and}$ is a conjunction in logic and Mathematics which gives us the following formula by switching letters.

$$P(A\ and\ B) = P(B)P(A|B) \leftrightarrow P(B\ and\ A) = P(A)P(B|A)$$

And since $A\ and\ B\ \leftrightarrow B\ and\ A$ we obtain the formula known as the **Multiplication Rule.**

$$\boldsymbol{P(A\ and\ B) = P(A)P(B|A)}$$

The second Formula is known as **Bayes Law (Bayes Rule or Bayes Theorem)** that has its own branch of study known as Bayesian inference. It's extremely easy to deduce at this point so I will continue with it's development (proof).

Recall that we demonstrated that $P(A|B) = \frac{P(A\ and\ B)}{P(B)}$ and we know by the Multiplication Rule $P(A\ and\ B) = P(A)P(B|A)$ we can substitute to obtain **Bayes' Law**

$$\boldsymbol{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

We will look at **Bayes Law** in the future, but for now we will focus on the Multiplication Rule for Probability.

We use the **Multiplication Rule for Probability** when we are selecting more than one item and we make our selections one at a time.

Two Selections

$$P(A\ and\ B) = P(A)P(B|A)$$

A happens first and B happens second.

Since we are selecting items one at a time, an important concept to determine before we start a problem is whether our selections are **with replacement** or **without replacement**. I typical start this lecture with a bag of marble example. The problem will go somewhat like this.

Color	Number
Red	6
White	4
Blue	8
Green	2
Yellow	1
Black	5

If you select two marbles **without replacement** from this bag, what's the probability they are:

1. Both Red?
2. Both White?
3. Both Green?
4. Both Yellow?
5. Both Non-Blue?
6. Both Non-Black?
7. Both Non-White?
8. Both Non-Yellow?

If you select two marbles **with replacement** from this bag, what's the probability they are:

9. Both Red?
10. Both White?
11. Both Green?
12. Both Yellow?
13. Both Non-Blue?
14. Both Non-Black?
15. Both Non-White?
16. Both Non-Yellow?

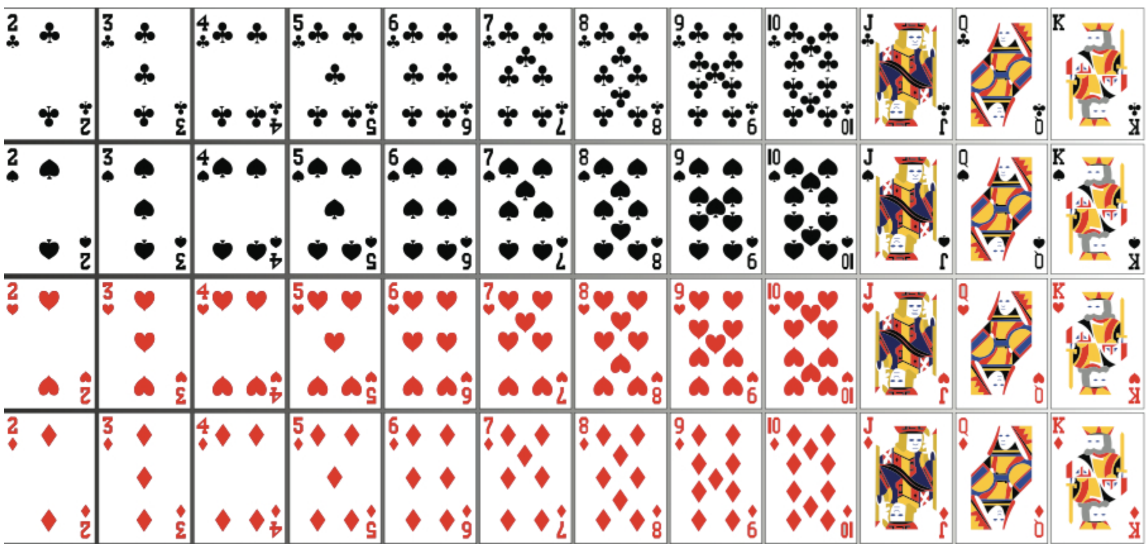
We can generalize this process and consider three selections and our formula will look like this.

Three Selections

$$P(A \text{ and } B \text{ and } C) = P(A)P(B|A)P(C|A \text{ and } B)$$

A happens first, B happens second, and C happens third.

Standard Deck Assume the Ace is High



If you select three **different** cards at random, what's the probability they are:
17. All Kings?
18. All Hearts?
19. All Red Cards?
20. All Face Cards?
21. All cards less than 4?
22. None are Kings?
23. None are Hearts?
24. None are Face Cards?
25. None are less than 4?

If you select three cards at random **with replacement**, what's the probability:

26. All Kings?
27. All Hearts?
28. All Red Cards?
29. All Face Cards?
30. All cards less than 4?
31. None are Kings?
32. None are Hearts?
33. None are Face Cards?
34. None are less than 4?

Fact- $P(A|B) \neq P(B|A)$
rarely are these probabilities are equal.

Def-Independent Events A and B A and B are independent, if $P(A) = P(A|B)$

That is the condition B does not change the likelihood of A. B has no effect on A.

Def-Dependent Events A and B A and B are dependent, if $P(A) \neq P(A|B)$

That is the condition B does change the likelihood of A. B has an effect on A.

These definitions will be of some value when we compute probabilities for multiple selections.