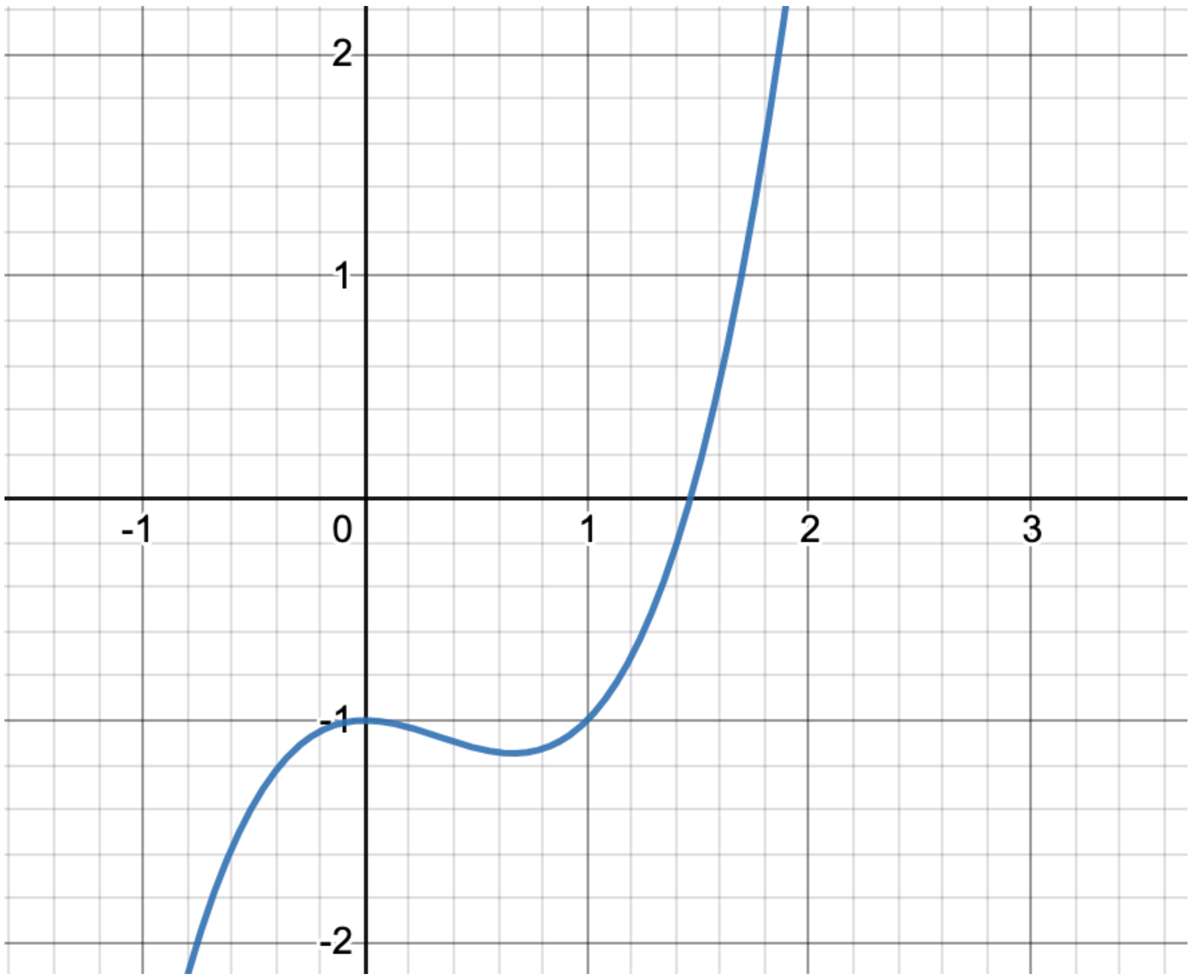
Newton's Method

Powerful numerical approximation method that is used to solve equations.

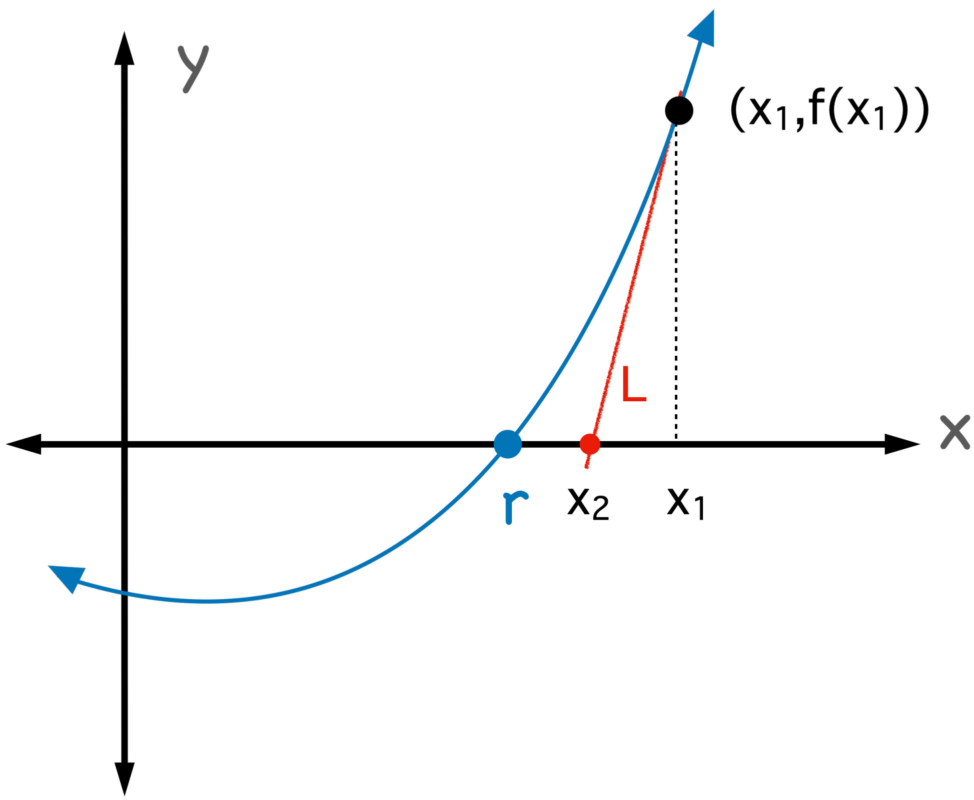
What algebra can we use to solve $x^3 - x^2 = 1$?



Generally, there may be no algebra, or equations, that can be used to solve for x. What can we do? **Use Newton's Method**

Idea

Let $f(r) = 0$ where r is a root to the equation and make a guess for a solution to the equation x_1. Note the tangent line L to the curve at $(x_1, f(x_1))$ and call the x-intercept x_2 The idea behind Newton's Method is that the tangent line approximates the curve $f(x)$ and so it's x-intercept x_2 is "close" to the x-intercept r.



Formula for x_2

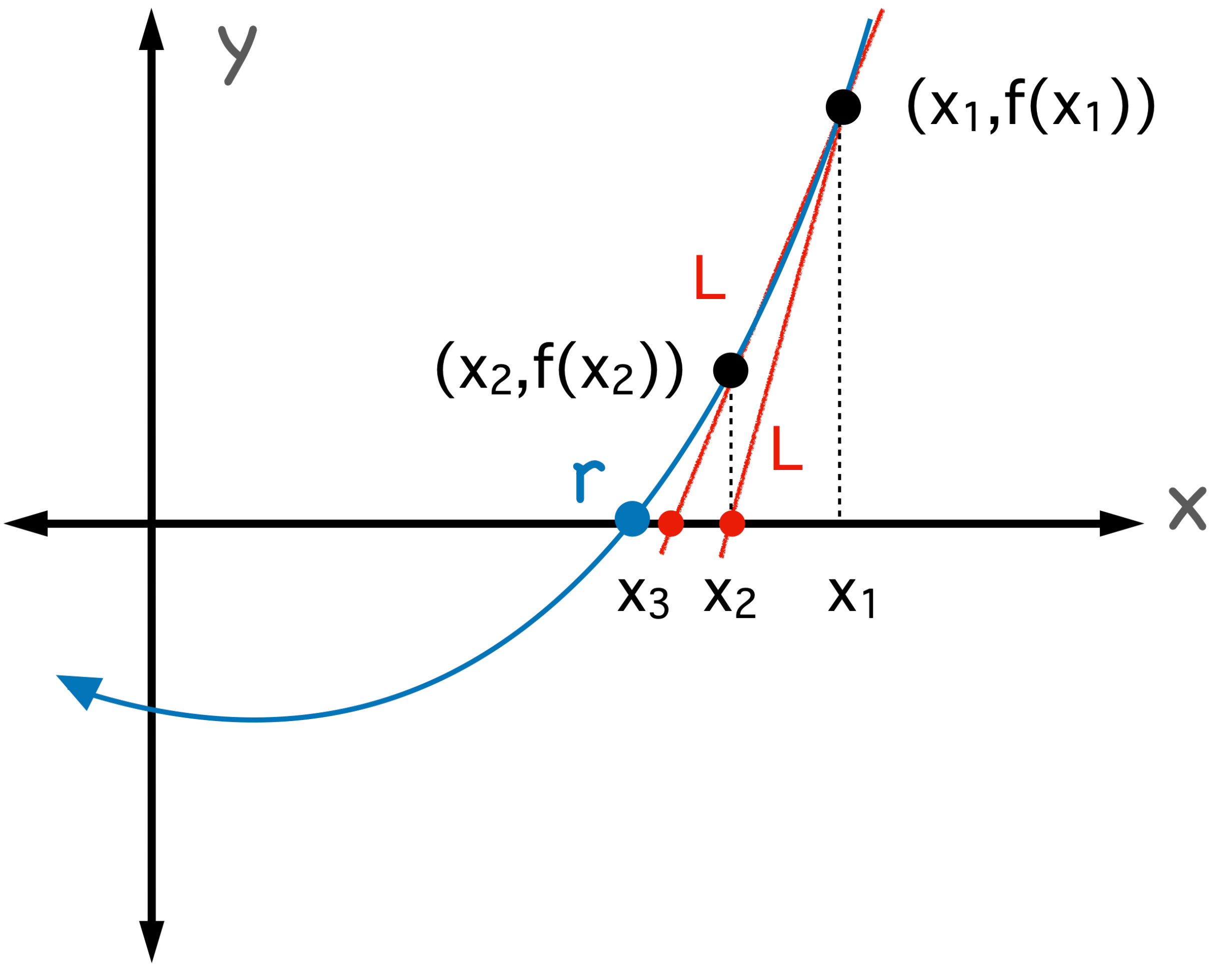
$$y - f(x_1) = f'(x_1)(x - x_1)$$

The x-intercept for L is $(x_2, 0)$

$0 - f(x_1) = f'(x_1)(x_2 - x_1)$ solving for x_2

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ assuming $f'(x_1) \neq 0$

Use x_2 as a second approximation for r



Formula for x_3

$$y - f(x_2) = f'(x_2)(x - x_2)$$

The x-intercept for L is $(x_3, 0)$

$0 - f(x_2) = f'(x_2)(x_3 - x_2)$ solving for x_3

$x_3 = x_2 - \frac{f(x_2)}{f\prime(x_2)}$ assuming $f'(x_2) \neq 0$

Keep repeating the process and obtain a **sequence** of approximations $x_1, x_2, x_3, x_4, \ldots$ to the solution r. Generally, we have the nth approximation.

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \text{ assuming } f'(x_{n-1}) \neq 0$$

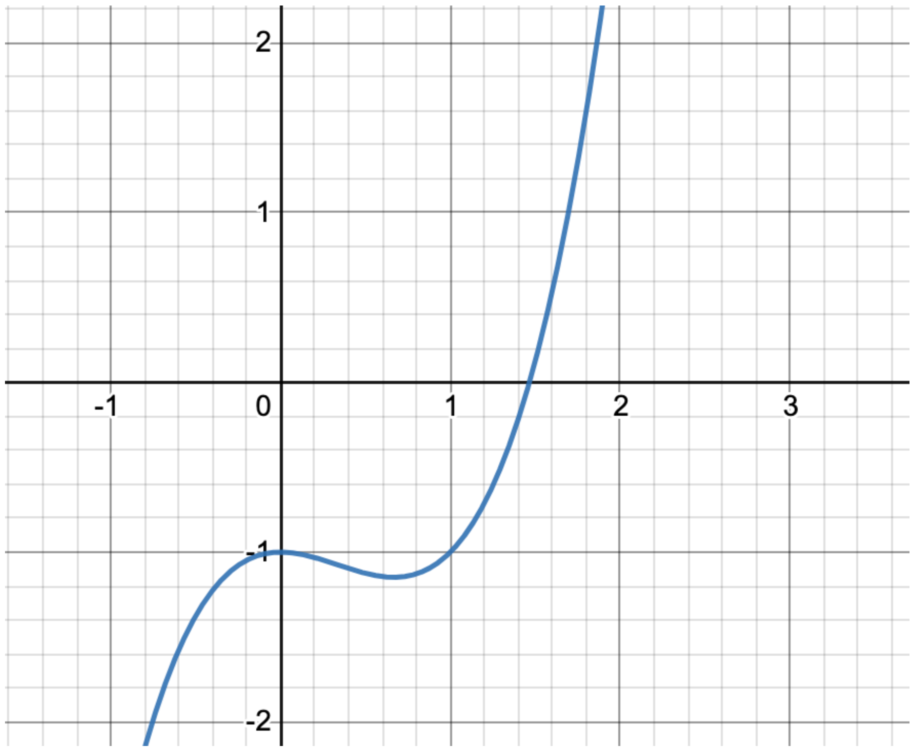
And, the numbers x_n get closer and closer to r as n gets big. That is, the **sequence** $x_1, x_2, x_3, x_4, \ldots$ converges to r.

$$\lim_{n \to \infty} x_n = r$$

Caution- Newton's Method does not always work for your initial guess x_1. If Newton's Method fails, make another guess for x_1.

Example

Solve $x^3 - x^2 = 1$ using Newton's Method with $x_1 = 1$



Let $f(x) = x^3 - x^2 - 1$ so that the solution will be a root r for $f(x)$.

$$f'(x) = 3x^2 - 2x$$

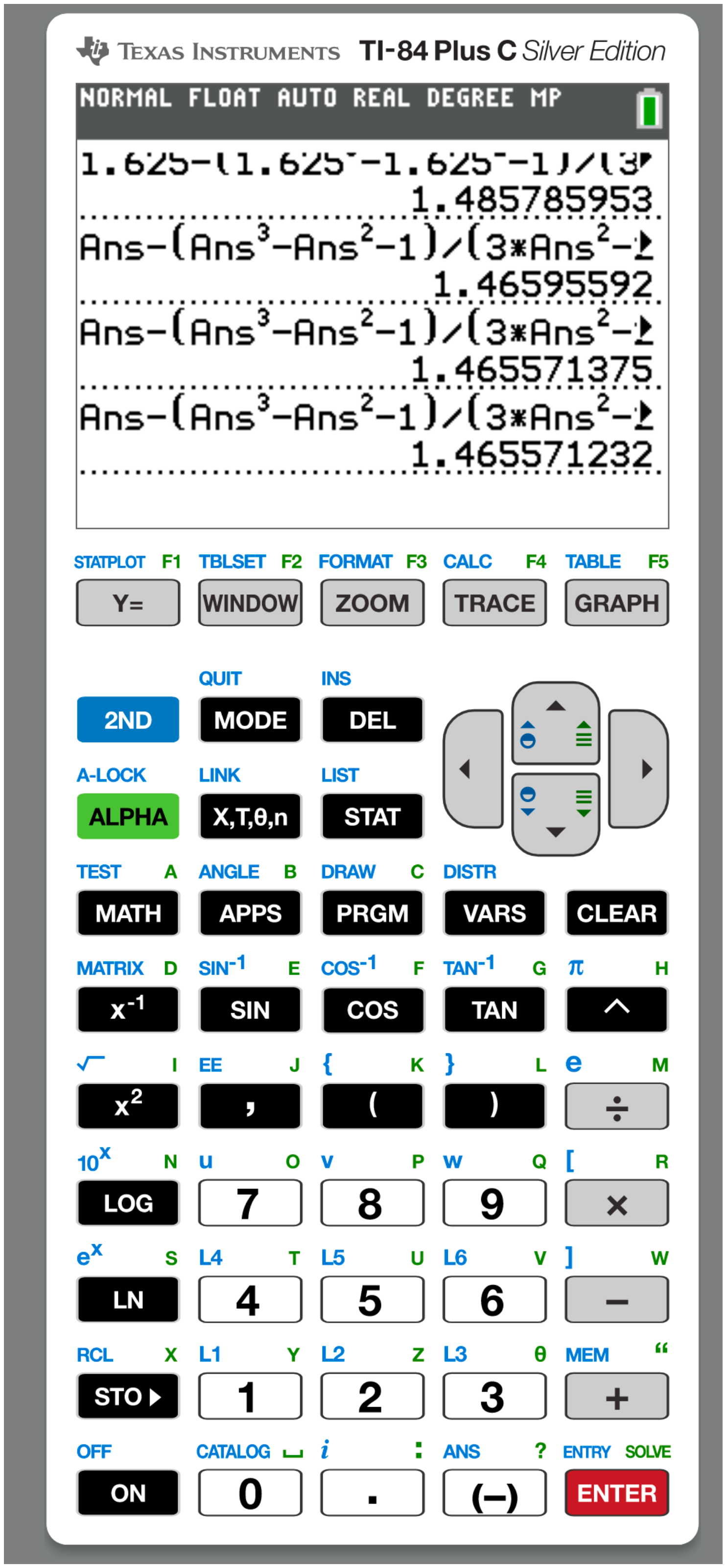
nth general term

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \text{ assuming } f'(x_{n-1}) \neq 0$$

$$x_n = x_{n-1} - \frac{x_{n-1}^3 - x_{n-1}^2 - 1}{3x_{n-1}^2 - 2x_{n-1}} \text{ assuming } f'(x_{n-1}) \neq 0$$

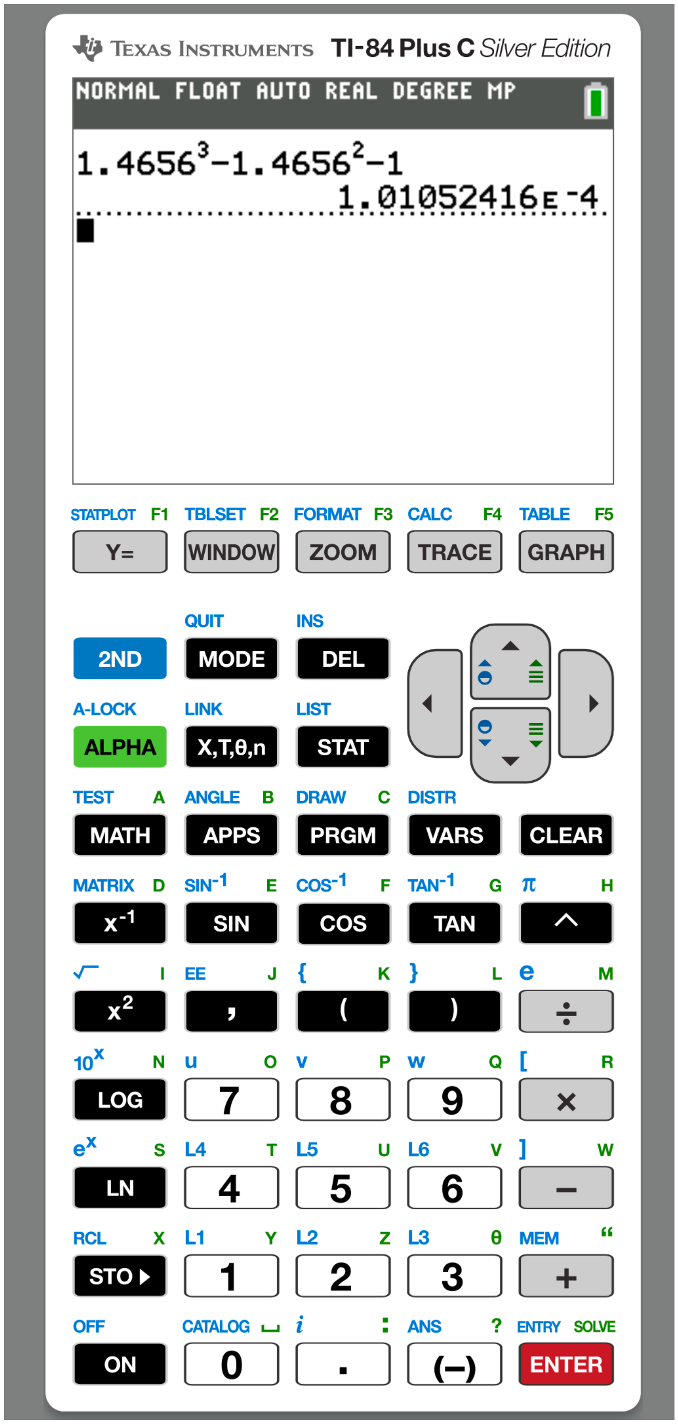
Let $x_1 = 1$

Then $x_2 = 2$ and $x_3 = 1.625$ and $x_4 \approx 1.4858$ and $x_5 \approx 1.4660$



$x_6 \approx 1.4656$ and $x_7 \approx 1.4656$

$$f(1.4656) \approx 0.0001$$



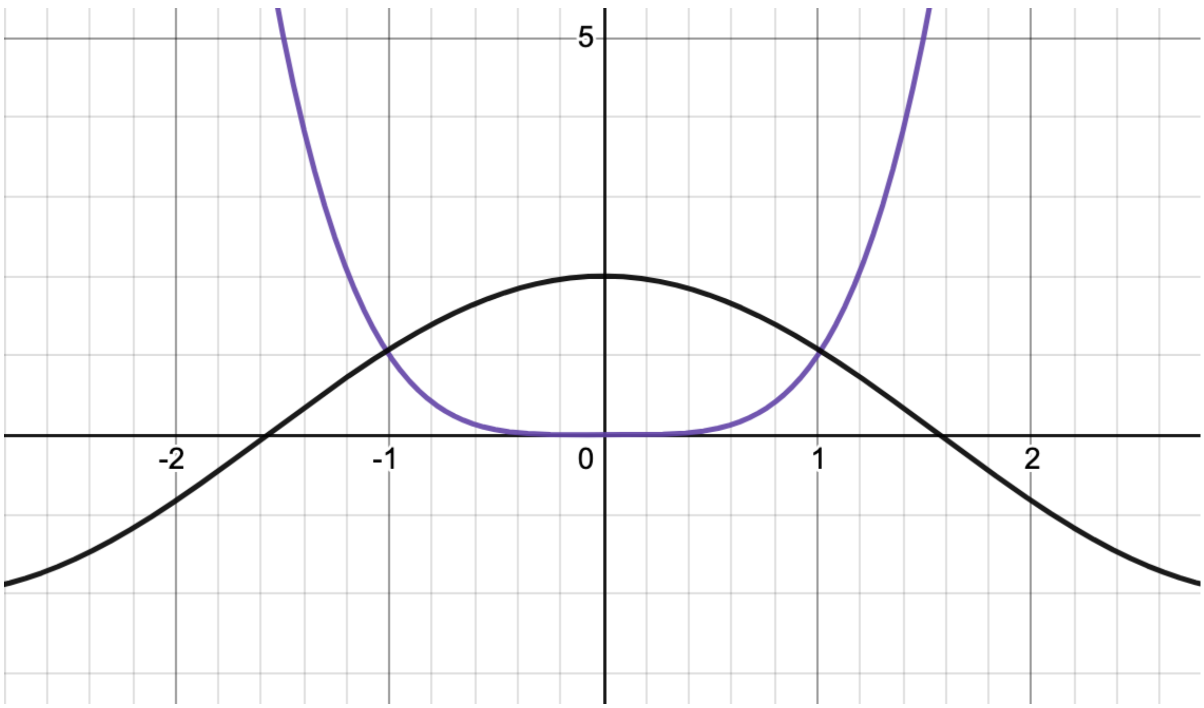
Thus $r \approx 1.4656$ is a root to the function $f(x) = x^3 - x^2 - 1$

Use Newton's Method to solve the following equation with a given initial guess x_1. **Approximate to 4 decimal places.**

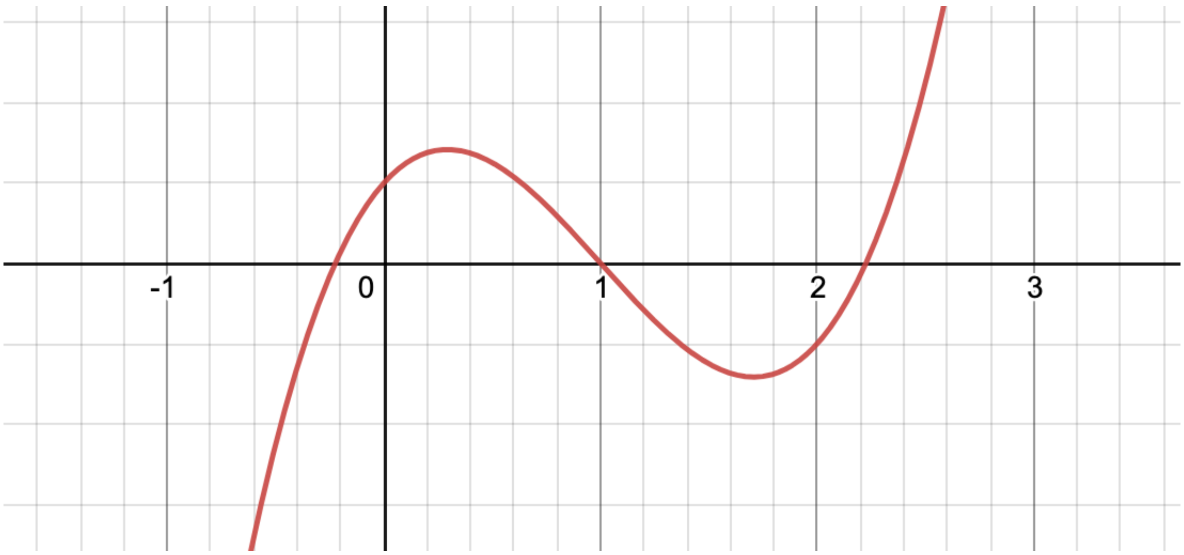
1. $x^5 + 2 = 0$ with $x_1 = -1$
2. $x^3 + 2x = 4$ with $x_1 = 1$

Use Newton's Method to solve the following equation when the root is in the given interval. **Approximate to 6 decimal places.**

3. $x^4 = 2\cos(x)$ with r in $[0,2]$

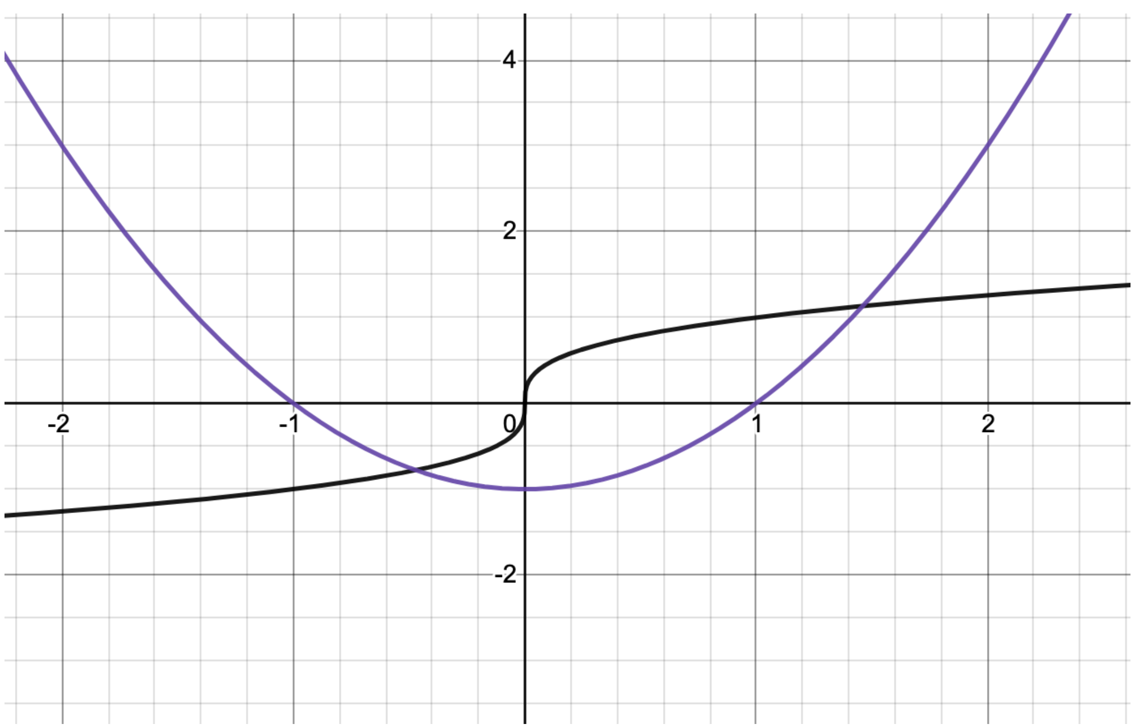


4. $2x^3 - 6x^2 + 3x + 1 = 0$ with r in $[2,3]$

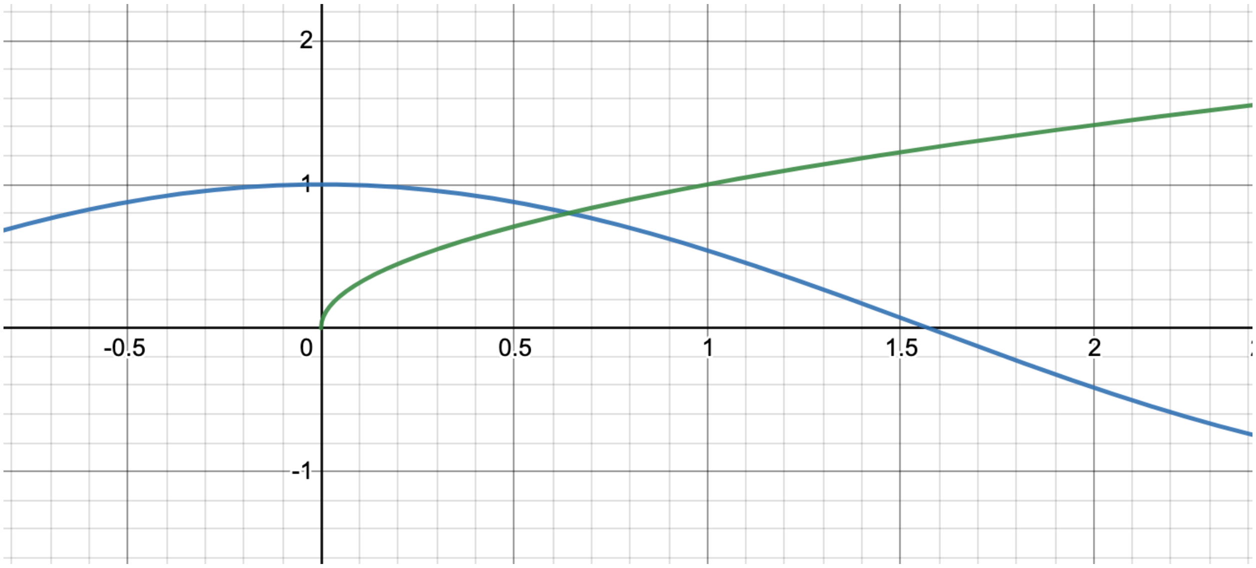


Use Newton's Method to find all solutions to the following equations.
Approximate to 6 decimal places.

5. $\sqrt[3]{x} = x^2 - 1$



6. $\cos(x) = \sqrt{x}$



Answers

1. $r = -1.1529$

2. $r = 1.1797$

3. $r = 1.013958$

4. $r = 2.224745$

5. $r = -0.471074$ and $r = 1.461070$

6. $r = 0.641714$