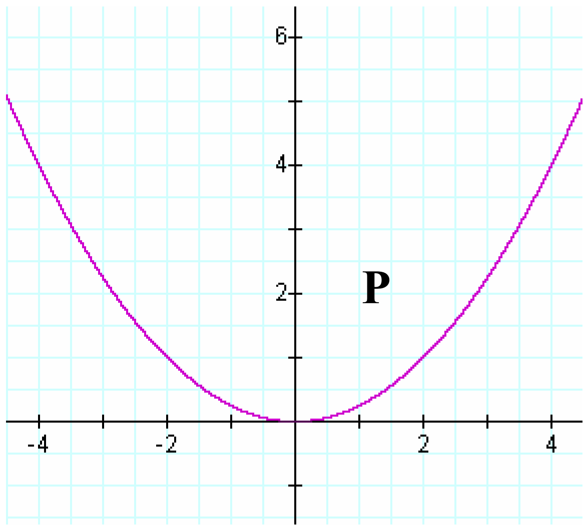
# Instantaneous Velocity and the Limit

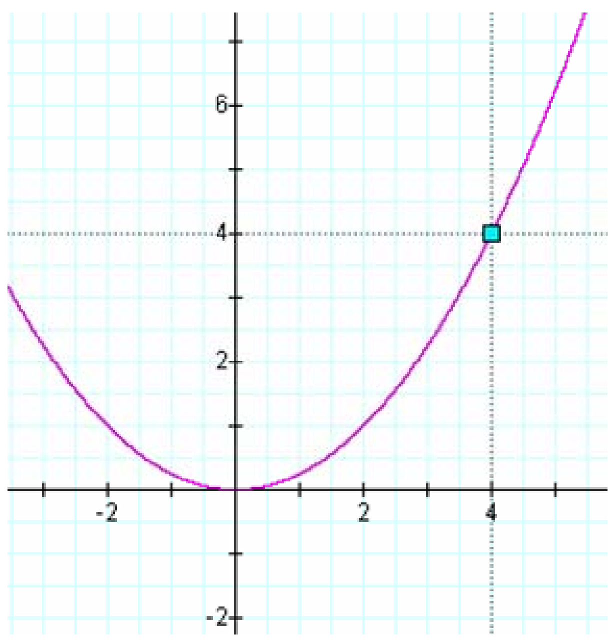
Using **secant lines** to approximate a **tangent line** (line tangent to a curve f at a point P) has an extremely exciting application in physical science. This process is used to approximate **instantaneous velocity** (velocity at a particular instant). It also illustrates another important concept in mathematics called a **limit**. These ideas are the founding principles behind what is known as **Differential Calculus**.

## Secant Lines

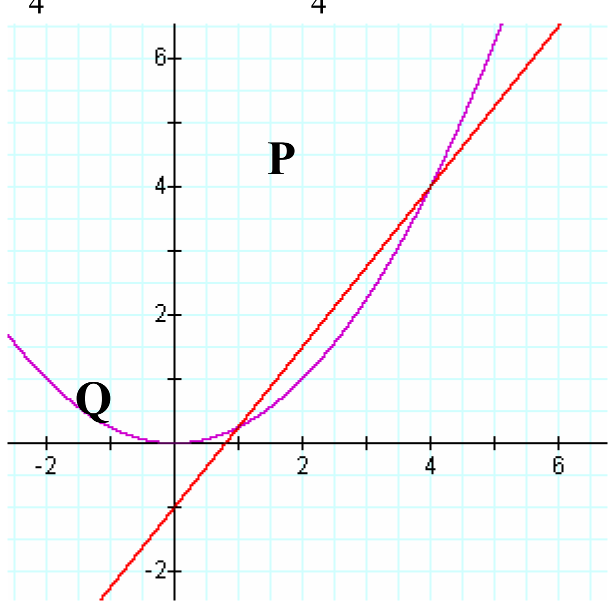
Let's consider the function $f(x) = \frac{x^2}{4}$ whose graph looks like the following.




Let's fix a point $P = (4,4)$ on the curve $f(x) = \frac{x^2}{4}$. It should look like the following.




Let's pick another point $Q = (1, \frac{1}{4})$ on the curve $f(x) = \frac{x^2}{4}$ and draw a line through these two points.



$\boxed{Q}$ ***What is the slope of the line through the points P and Q on the curve*** $f(x)=\frac{x^2}{4}$ ***?***

**Answer-** The equation of the slope is $m=\frac{y_2-y_1}{x_2-x_1}$ where $P=(4,4)_1$ and $Q=(1,\frac{1}{4})_2$

$$m=\frac{\frac{1}{4}-4}{1-4}=\frac{\frac{-15}{4}}{-3}=\frac{\frac{15}{4}}{\frac{3}{1}}=\frac{15}{4}\cdot\frac{1}{3}=\frac{5}{4}=1.25$$

The slope of the line through points P and Q is 1.25.

$\boxed{Q}$ ***What does this have to do with velocity?***

**Answer-** Consider the distance function s(t) = the distance a particle travels after time t has elapsed. If $s(t)=\frac{t^2}{4}$ where t represents time (in seconds) and s represents distance traveled (in feet), then the following is known.

<u>The distance the particle traveled after 1 second is 0.25 feet.</u>

$$s(1)=\frac{1^2}{4}=\frac{1}{4}=0.25$$

<u>The distance the particle traveled after 4 seconds is 4 feet.</u>

$$s(4)=\frac{4^2}{4}=\frac{16}{4}=4$$

<u>Between the 1st and 4th seconds, the particle traveled 3.75 feet.</u>

$$s(4)-s(1)=4-0.25=3.75$$

<u>The average velocity during this time interval of 3 seconds is 1.25 feet/sec.</u>

$$velocity=\frac{\Delta s}{\Delta t}=\frac{s(4)-s(1)}{4-1}=\frac{3.75}{3}=1.25$$

The **slope** of the secant line through points P and Q is the **average velocity** of the particle between t=1 and t=4 when the distance function is $s(t)=\frac{t^2}{4}$

$\boxed{Q}$ ***What does this have to do with instantaneous velocity?***

**Answer-** Let's say we are interested in obtaining the instantaneous velocity of the particle at t=4 seconds. All we need to do is obtain better, and better, approximations to the average velocity. That is, we need to determine the slopes of the secant lines through the following points.

| P | | Q | | |
|---|---|---|---|---|
| x | y | x | y | m |
| 4 | 4 | 1 | 0.25 | 1.25 |
| 4 | 4 | 2 | 1 | 1.5 |
| 4 | 4 | 3 | 2.25 | 1.75 |
| 4 | 4 | 3.5 | 3.0625 | 1.875 |
| 4 | 4 | 3.7 | 3.4225 | 1.925 |
| 4 | 4 | 3.9 | 3.8025 | 1.975 |
| 4 | 4 | 3.99 | 3.980025 | 1.9975 |
| 4 | 4 | 3.999 | 3.998 | 1.99975 |
| 4 | 4 | 3.9999 | 3.9998 | 1.999975 |

$$y=\frac{x^2}{4}$$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$m=\frac{\frac{x_2{}^2}{4}-4}{x_2-4}$$

**This concept is known as taking the limit. As x approaches 4 from the left, the average velocities approach the "instantaneous" velocity of <u>2 feet per second</u>.**

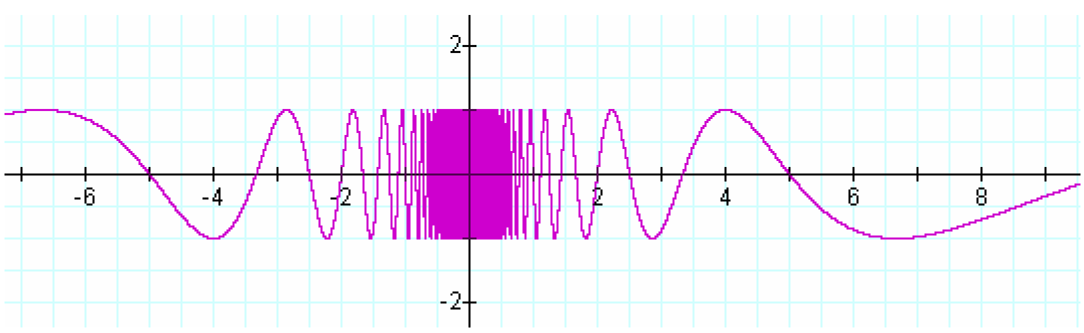
**The secant lines through points P and Q become a tangent line as Q approaches P on the curve f.**

**Q** ***Let*** $s(t) = 40t - 16t^2$ ***be the distance a particle traveled in feet after t seconds. Calculate the instantaneous speed at t=2 seconds.***
**Answer-** Construct a table of secant lines through the point P=(2,16).

| P | | Q | | |
|---|---|---|---|---|
| x | y | x | y | m |
| 2 | 16 | 1 | 24 | -8 |
| 2 | 16 | 1.5 | 24 | -16 |
| 2 | 16 | 1.6 | 23.04 | -17.6 |
| 2 | 16 | 1.7 | 21.76 | -19.2 |
| 2 | 16 | 1.8 | 20.16 | -20.8 |
| 2 | 16 | 1.9 | 18.24 | -22.4 |
| 2 | 16 | 1.99 | 16.2384 | -23.84 |
| 2 | 16 | 1.999 | 16.02398 | -23.984 |
| 2 | 16 | 1.9999 | 16.0024 | -23.9984 |

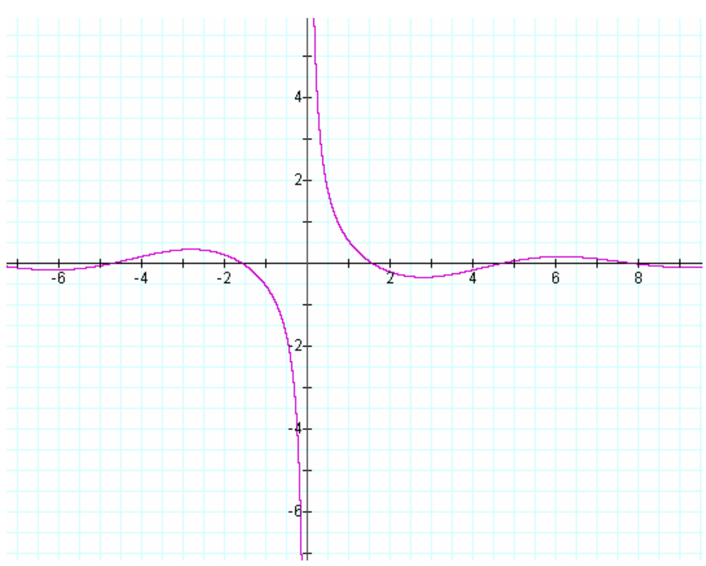
**Q** ***Let*** $f(x) = \sin(\frac{10\pi}{x})$ ***and point P=(1,0) on the curve f. For the following values of Q, determine the slope of the secant lines through the points P and Q on the curve f.***
**Answer-** Numerical and Graphical.

| P | | Q | | |
|---|---|---|---|---|
| x | y | x | y | m |
| 1 | 0 | 2 | 0.0002 | 0.0002 |
| 1 | 0 | 1.5 | 0.8661 | 1.7323 |
| 1 | 0 | 1.4 | -0.4337 | -1.0842 |
| 1 | 0 | 1.3 | -0.8231 | -2.7438 |
| 1 | 0 | 1.2 | 0.8659 | 4.3294 |
| 1 | 0 | 1.1 | -0.2814 | -2.8145 |
| 1 | 0 | 0.5 | -0.0007 | 0.0013 |
| 1 | 0 | 0.6 | 0.8663 | -2.1657 |
| 1 | 0 | 0.7 | 0.7815 | -2.6051 |
| 1 | 0 | 0.8 | 1.0000 | -5.0000 |
| 1 | 0 | 0.9 | -0.3417 | 3.4168 |



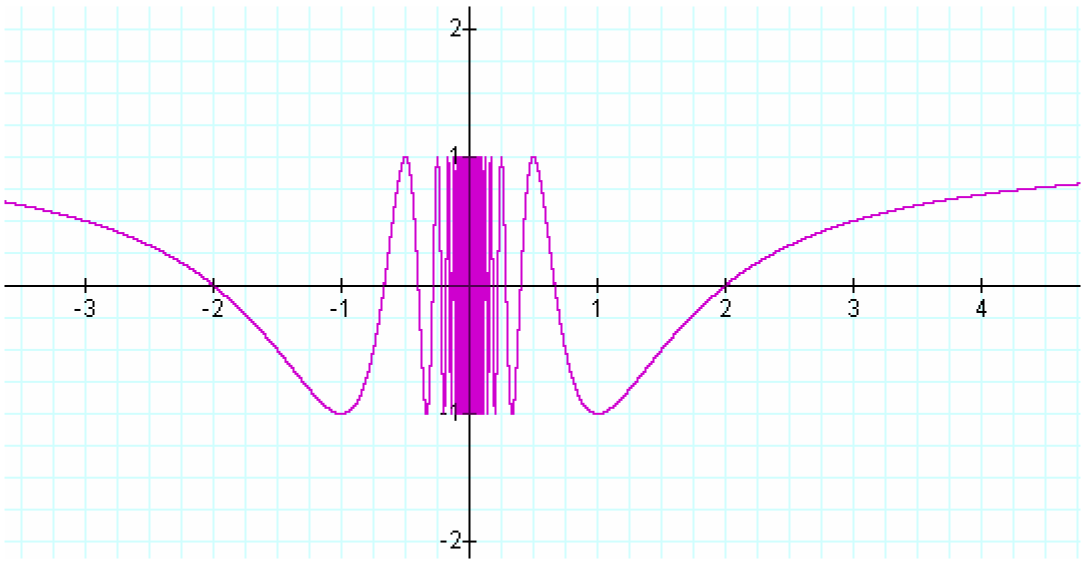

**Q** ***Let*** $f(x) = \frac{\cos(x)}{x}$. ***Determine the limit as x approaches 0 from the right and the left.***
***Answer-*** Numerical and Graphical

| x | y | x | y |
|---|---|---|---|
| 1 | 0.540302 | -1 | -0.5403 |
| 0.5 | 1.755165 | -0.5 | -1.75517 |
| 0.4 | 2.302652 | -0.4 | -2.30265 |
| 0.3 | 3.184455 | -0.3 | -3.18445 |
| 0.2 | 4.900333 | -0.2 | -4.90033 |
| 0.1 | 9.950042 | -0.1 | -9.95004 |
| 0.01 | 99.995 | -0.01 | -99.995 |
| 0.001 | 999.9995 | -0.001 | -1000 |
| 0.0001 | 10000 | -0.0001 | -10000 |
| 0.00001 | 100000 | -0.00001 | -100000 |



**Q** ***Let*** $f(x)=\cos\left(\frac{\pi}{x}\right)$. ***Determine the limit as x approaches 0 from the right and the left.***

**Answer-** Numerical and Graphical.

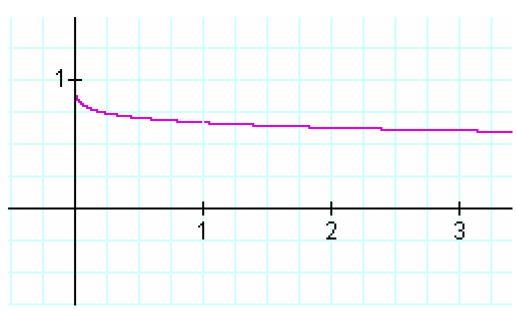



| x | y | x | y |
|---|---|---|---|
| 1 | -1 | -1 | -1 |
| 0.5 | 1 | -0.5 | 1 |
| 0.4 | 8.16E-05 | -0.4 | 8.16E-05 |
| 0.3 | -0.50009 | -0.3 | -0.50009 |
| 0.2 | -1 | -0.2 | -1 |
| 0.1 | 1 | -0.1 | 1 |
| 0.01 | 0.999995 | -0.01 | 0.999995 |
| 0.001 | 0.999467 | -0.001 | 0.999467 |
| 0.0001 | 0.947159 | -0.0001 | 0.947159 |
| 0.00001 | -0.99235 | -0.00001 | -0.99235 |
| 0.000001 | 0.327005 | -0.000001 | 0.327005 |

**Q** ***Let*** $f(x)=\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$. ***Determine the limit as x approaches 1 from the right and the left.***

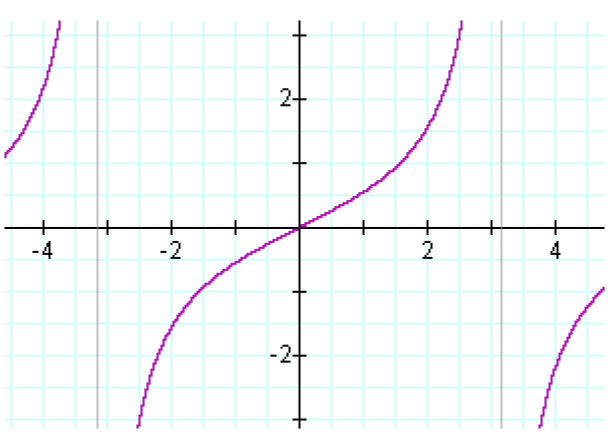
**Answer-** Numerical and Graphical

| x | y | x | y |
|---|---|---|---|
| 2 | 0.627505 | 0 | 1 |
| 1.9 | 0.630442 | 0.1 | 0.783654 |
| 1.7 | 0.636792 | 0.2 | 0.751097 |
| 1.5 | 0.643905 | 0.3 | 0.730894 |
| 1.3 | 0.651989 | 0.5 | 0.70435 |
| 1.1 | 0.661358 | 0.7 | 0.686274 |
| 1.01 | 0.666114 | 0.9 | 0.672503 |
| 1.001 | 0.666611 | 0.99 | 0.667225 |
| 1.0001 | 0.666661 | 0.999 | 0.666722 |
| 1.00001 | 0.666666 | 0.9999 | 0.666672 |




**Q** ***Let*** $f(x)=\frac{\cos(x)-1}{\sin x}$. ***Determine the limit as x approaches 0 from the right and the left.***

**Answer-** Numerical and Graphical




| x | y | x | y |
|---|---|---|---|
| 1 | -0.5463 | -1 | 0.546302 |
| 0.5 | -0.25534 | -0.5 | 0.255342 |
| 0.3 | -0.15114 | -0.3 | 0.151135 |
| 0.1 | -0.05004 | -0.1 | 0.050042 |
| 0.01 | -0.005 | -0.01 | 0.005 |
| 0.001 | -0.0005 | -0.001 | 0.0005 |
| 0.0001 | -5E-05 | -0.0001 | 5E-05 |