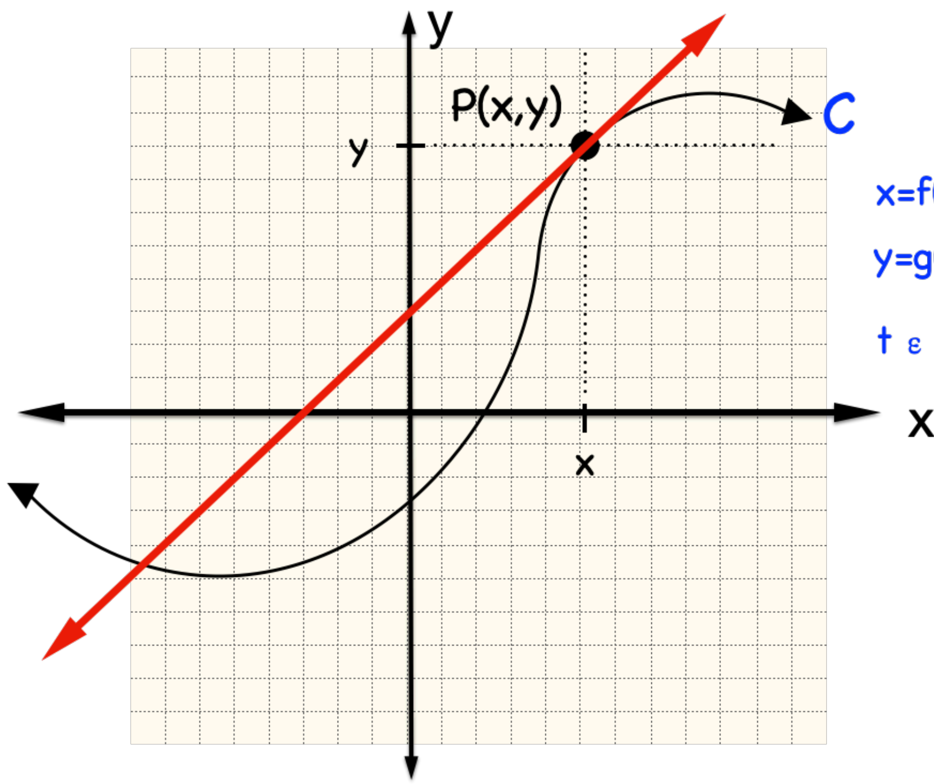
Calculus of Parametric Equations

Let C represent a curve in Parametric Form (Parametric Equations)

$$x = f(t)$$
$$y = g(t)$$
$$t \in I$$

we would like to determine the derivative of the function of x that is described by the curve C parametrically.



Note- We can't use $y = f(x)$ as f is used in describing the variable x. Thus, we really would like to know what is dy/dx?

$$\frac{dy}{dx} = \frac{d}{dx}[g(t)] = g'(t)\frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ by Chain Rule}$$

That is,

Derivative

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ assuming } \frac{dx}{dt} \neq 0$$

The second derivative can be found similarly, but I will let $\frac{dy}{dx} = u(t)$ as it is a function of t.

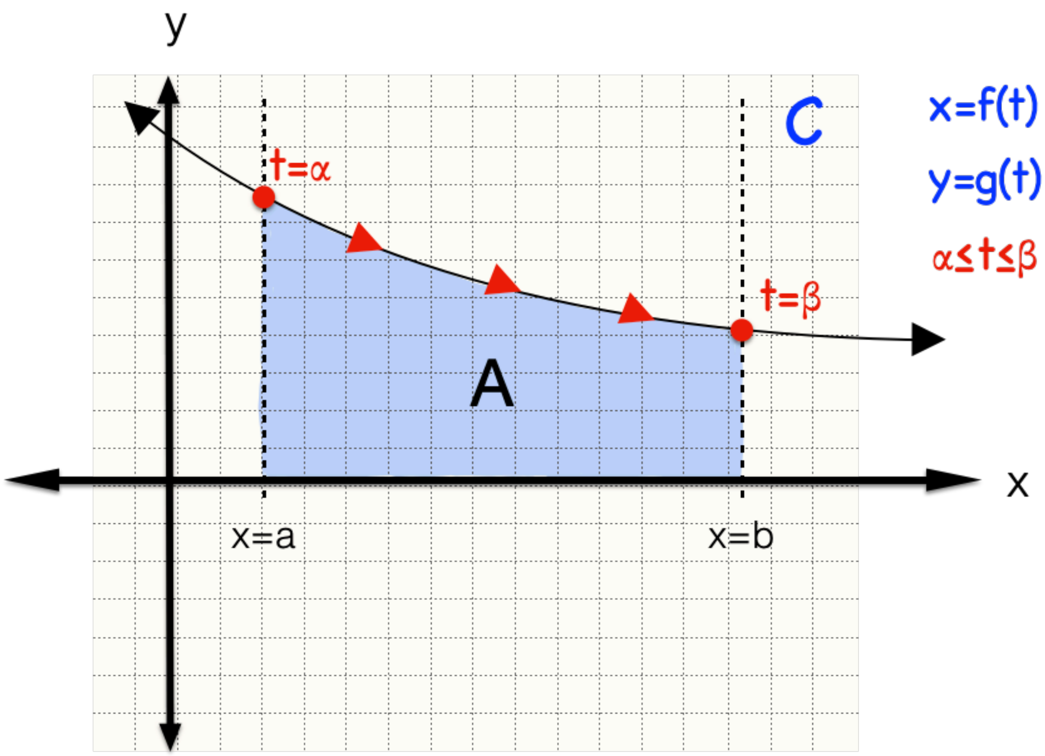
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[u(t)] = \frac{du}{dt} \cdot \frac{dt}{dx} = \frac{\frac{du}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad \text{by Chain Rule}$$

That is,

Second Derivative

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Area Under a Curve



The curve C is above the x-axis

The curve C is traversed (move through) once as t increases from α to β, $\boldsymbol{\alpha \leq t \leq \beta}$

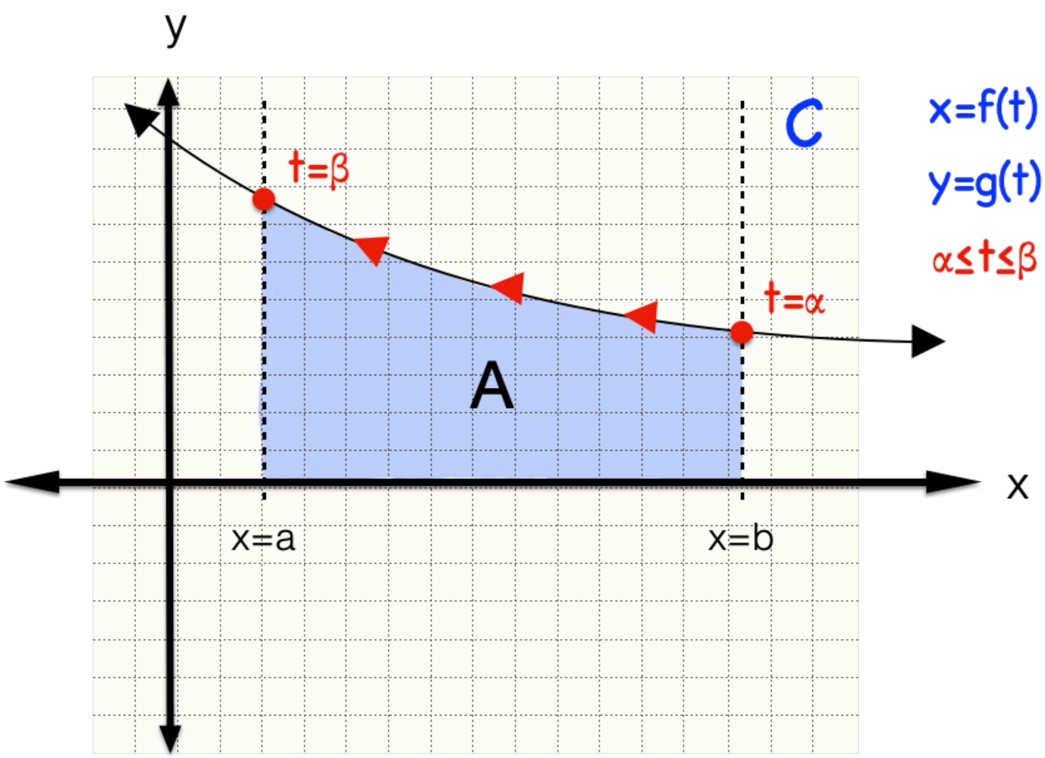
x is increasing, so $\frac{dx}{dt} > 0$

Substitute

$$A = \int_{x=a}^{x=b} y dx = \int_{t=\alpha}^{t=\beta} g(t) f'(t) dt$$

Note- What happens when x is decreasing over $\boldsymbol{\alpha} \leq \boldsymbol{t} \leq \boldsymbol{\beta}$?

That is, $\frac{dx}{dt} < 0$?

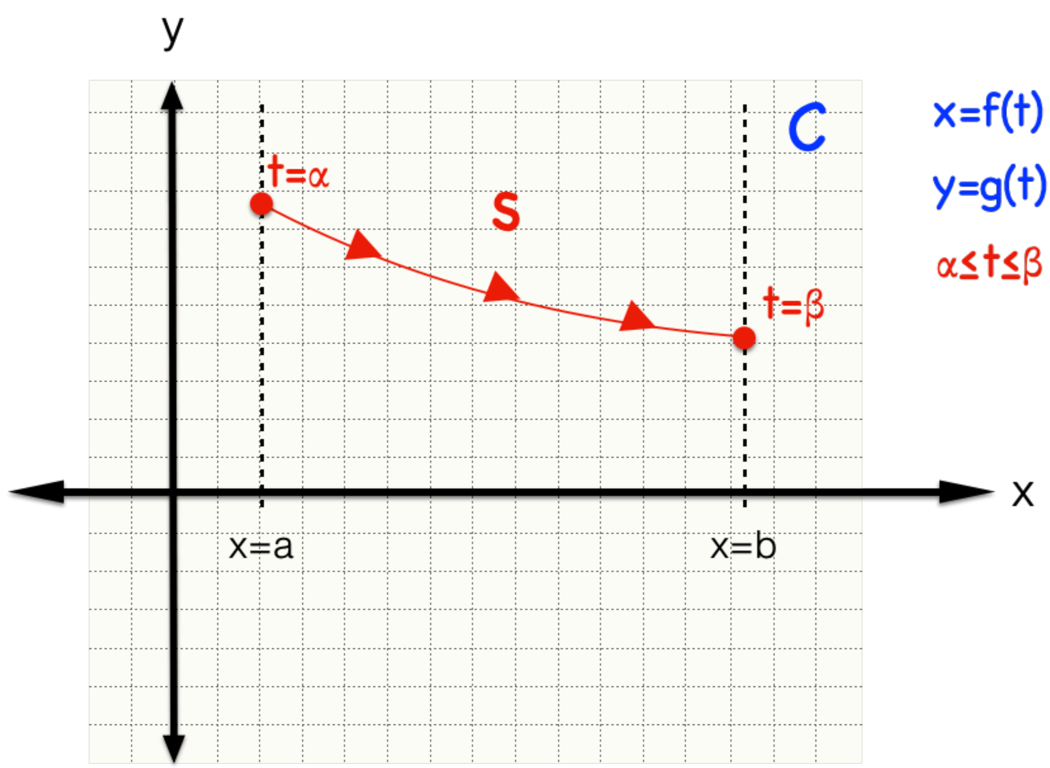


Substitute

$$A = \int_{x=a}^{x=b} y dx = \int_{t=\beta}^{t=\alpha} g(t) f'(t) dt = -\int_{t=\alpha}^{t=\beta} g(t) f'(t) dt$$

Arc Length

Let C be a curve defined be Parametric Equations $(x = f(t), y = g(t), \alpha \leq t \leq \beta)$ where f' and $g'$are continuous for $\alpha \leq t \leq \beta$ and C is tranverse exactly once as t increases from α to β. Then, we can determine the arc length of the curve from the initial point to the terminal point.



Recall the arc length formula where $\boldsymbol{S} = \int \boldsymbol{ds}$

$$S = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

Assuming $\frac{dx}{dt} > 0$ as illustrated

$$S = \int_{x=a}^{x=b} \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_{t=\alpha}^{t=\beta} \sqrt{1 + \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]^2}\, dx$$

$$= \int_{t=\alpha}^{t=\beta} \sqrt{\left[\frac{\frac{dx}{dt}}{\frac{dx}{dt}}\right]^2 + \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]^2}\, dx = \int_{t=\alpha}^{t=\beta} \frac{\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}}{\sqrt{\left[\frac{dx}{dt}\right]^2}}\, dx$$

$$= \int_{t=\alpha}^{t=\beta} \frac{\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}}{\left|\frac{dx}{dt}\right|}\, dx = \int_{t=\alpha}^{t=\beta} \frac{\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}}{\frac{dx}{dt}}\, dx$$

$$= \int_{t=\alpha}^{t=\beta} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} \cdot \frac{dt}{dx}\, dx = \int_{t=\alpha}^{t=\beta} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

$$\boxed{S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Note- Assuming $\frac{dx}{dt} < 0$ or x is decreasing

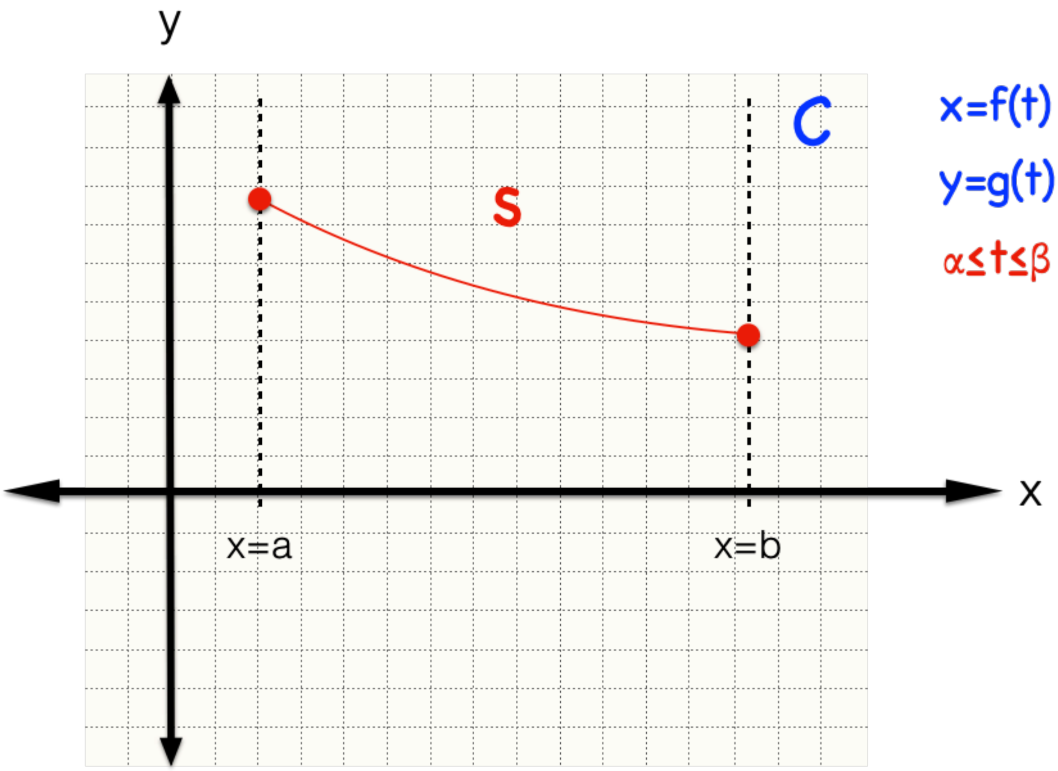
$$S = \int_{x=a}^{x=b} \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

$$= \int_{t=\beta}^{t=\alpha} \frac{\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}}{\left|\frac{dx}{dt}\right|}\, dx = \int_{t=\beta}^{t=\alpha} \frac{\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}}{-\frac{dx}{dt}}\, dx$$

$$= -\int_{t=\beta}^{t=\alpha} \frac{\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}}{\frac{dx}{dt}}\, dx = \int_{t=\alpha}^{t=\beta} \frac{\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}}{\frac{dx}{dt}}\, dx$$ switching limits

$$= \int_{t=\alpha}^{t=\beta} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} \cdot \frac{dt}{dx}\, dx = \int_{t=\alpha}^{t=\beta} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

Orientation does not make a difference when it comes to arc length!



$$S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Surface Area

$$\boldsymbol{SA = \int 2\pi r ds}$$

Let C be a curve defined be Parametric Equations ($x = f(t), y = g(t), \alpha \leq t \leq \beta$) where f' and $g'$are continuous for $\alpha \leq t \leq \beta$ and C is tranverse exactly once as t increases from α to β and assuming $g(t) \geq 0$.

Since $s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ we have $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Our Surface Area formulas hold up.

Rotate a Parametric Curve C about the x-axis

$$SA = \int_\alpha^\beta 2\pi r \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Rotate a Parametric Curve C about the y-axis

$$SA = \int_\alpha^\beta 2\pi r \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$