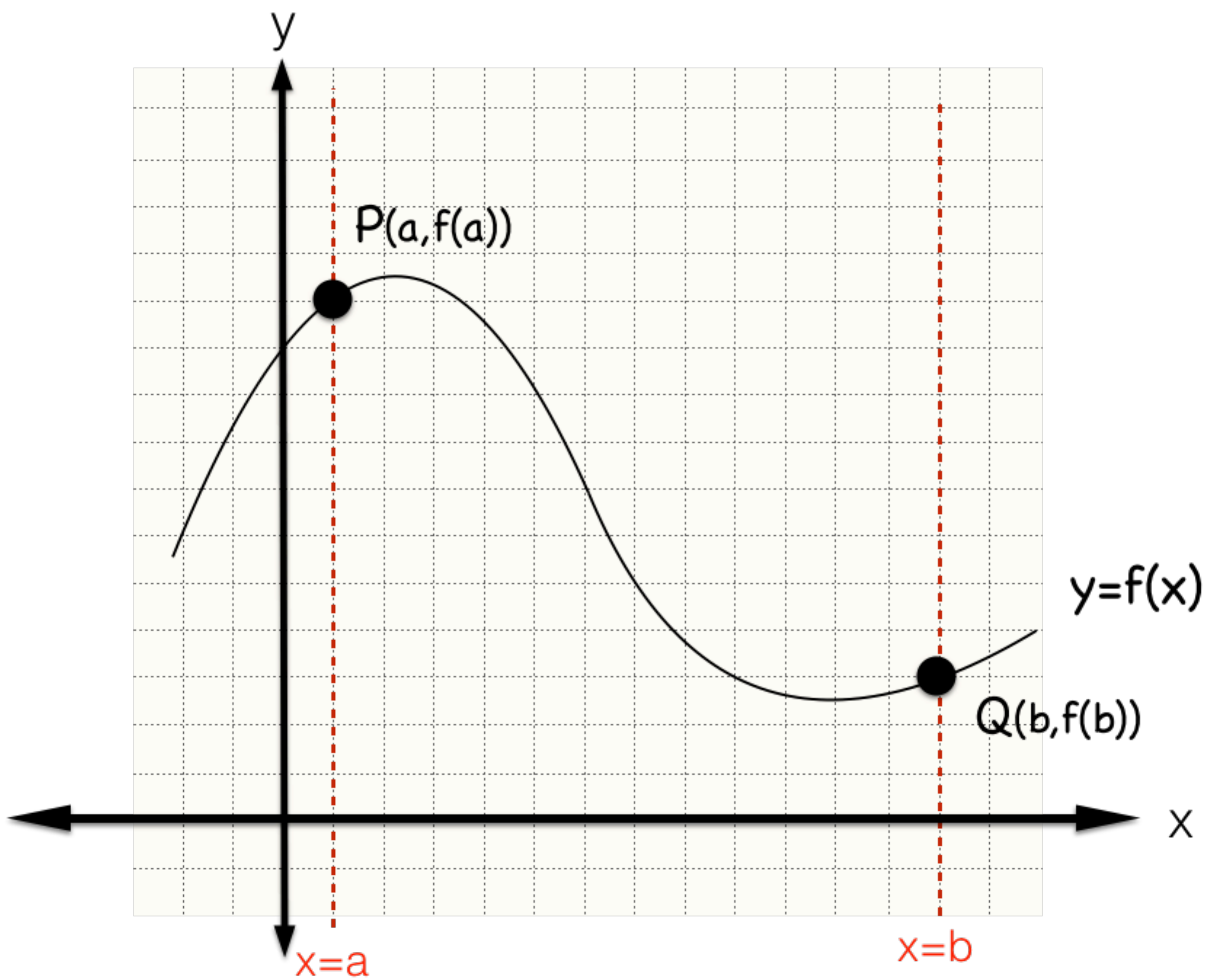
Arc Length s

Is the length along a smooth curve from one point to another. The assumptions are that the function $y = f(x)$ is continuous over $[a, b]$ and differentiable over (a, b) as well.

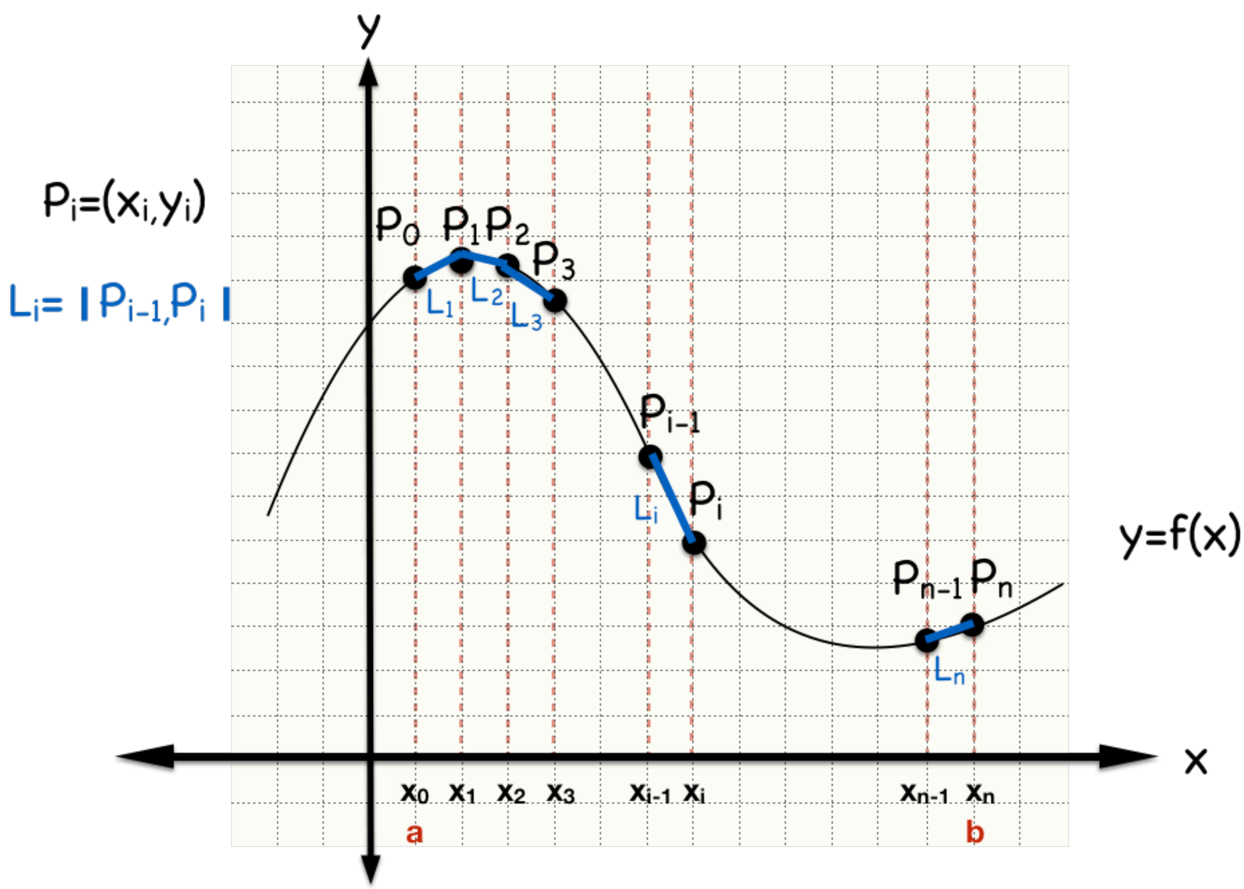


Assuming $f'(x)$ is continuous over $a \leq x \leq b$. The distance along the curve $y = f(x)$ from point P to point Q is represented by the following formula.

$$s = \int_a^b \sqrt{1 + [\,f'(x)]^2}\, dx \quad \text{or} \quad s = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

A proof of this formula starts by partitioning the interval $[a, b]$ into n-subintervals of equal length Δx in the same way we looked at numerical approximations.

These partitions are used to define points on our curve P_i so that we can define lengths between these points $L_i = |P_{i-1}P_i|$ for $i = 1\ to\ n$ as seen below. And, we can use the distance formula in Geometry to define $|P_{i-1}P_i|$ with some Algebra and the Mean Value Theorem.



The following is the proof of the Arc Length Formula. Since,

$L_i = |P_{i-1}P_i|$

$= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$ Distance Formula from Geometry

$= \sqrt{(x_i - x_{i-1})^2 + ((f(x_i) - f(x_{i-1})))^2}$ as $y = f(x)$

$= \sqrt{(x_i - x_{i-1})^2 + \left[f'(x_i^*)(x_i - x_{i-1})\right]^2}$ The Mean Value Theorem

$= \sqrt{[\Delta x]^2 + \left[f'(x_i^*)\Delta x\right]^2}$ Since $\Delta x = x_i - x_{i-1}$

$= \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x$ Factoring out $[\Delta x]^2$; product rule for square roots

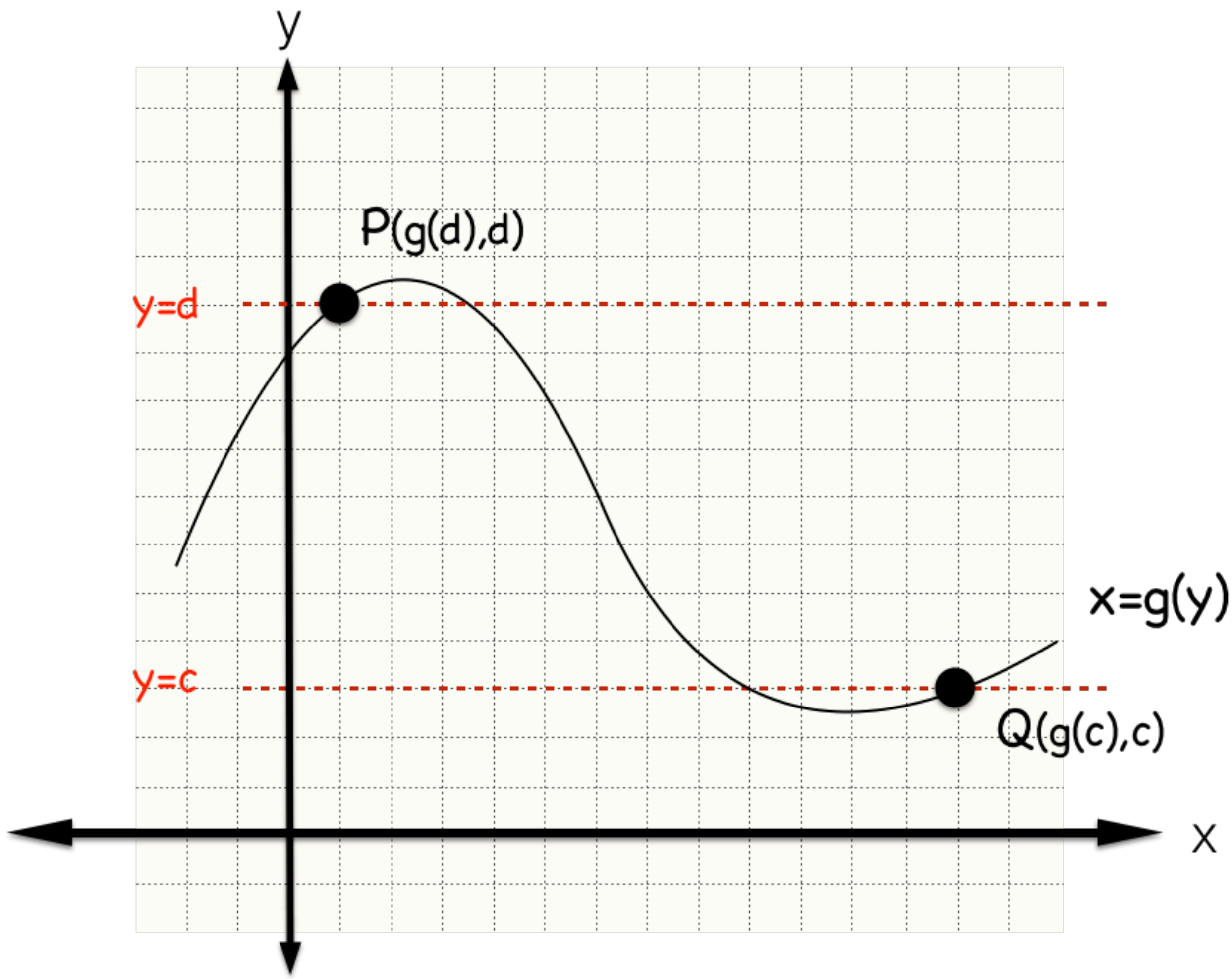
We can now approximate the distance from P to Q by summing the linear distance together represented by $L \approx L_1 + L_2 + \cdots + L_n = \sum_{i=1}^{n} L_i$. Now as $n \to \infty$ we no longer have an approximation we have an exact value via the Riemann Sum.

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

Instead of using the capital letter L to represent arc length, we denote it with the letter s to be consistant with the letter representation from Trigonometry $s = r\theta$.

$$\boxed{s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

Similarly, we can allow for a continuous smooth curve $x = g(y)$ over $c \leq y \leq d$ with $g'(y)$ being continuous over $c \leq y \leq d$. The arc length s along the curve $x = g(y)$ from P to Q.



$$s = \int_c^d \sqrt{1 + [\,g'(y)]^2}\,dy \quad \text{or} \quad s = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$$

Arc Length Function

When we look to determine the arc length of a smooth continuous curve from a fixed point $P(a, f(a))$ to an **arbitrary** $Q\big(x, f(x)\big)$ on a curve, we can create an arc length function by using a dummy variable t over the interval $[a, x]$ which means $a \leq t \leq x$.

$$s(x) = \int_a^x \sqrt{1 + [\,f'(t)]^2}\, dt$$

Now, by the Fundamental Theorem of Calculus part I, we know that we have the following derivative.

$$\frac{ds}{dx} = \sqrt{1 + [\,f'(x)]^2} \quad or \quad \frac{ds}{dx} = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}$$

Using some Algebra, we obtain a differential represented with the following.

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\; dx$$

Using some Algebra by squaring both sides, we obtain the following.

$$(ds)^2 = (dx)^2 + (dy)^2$$

Using some addition Algebra, and dividing both sides by $(dy)^2$ we can obtain the following equivalent differential.

$$ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\; dy$$

We will make great use of this differentials when we determine surface area for a solid of revolution.